A BESSEL FUNCTION INEQUALITY CONNECTED WITH STABILITY OF LEAST SQUARE SMOOTHING

LEE LORCH[1] AND PETER SZEGO

1. **Introduction.** In considering the stability (i.e., the asymptotic smoothness of higher order iterates) of continuous smoothing problems in least square approximation, H. S. Wilf [**7**(a)] introduces a certain inequality ((2) below) involving Bessel functions.

His argument in support of this inequality requires correction [**7**(b)], which it is our purpose here to supply.

2. **The inequality.** Defining

$$h_{\nu\lambda}(\theta) = 1 - \frac{\int_0^\theta t^{-\lambda} J_\nu(t)\,dt}{\int_0^\infty t^{-\lambda} J_\nu(t)\,dt}, \tag{1}$$

where $J_\nu(t)$ is the Bessel function of first kind and order ν, the inequality in question is

$$-1 < h_{\nu\lambda}(\theta) < 1 \qquad (\theta \neq 0), \tag{2}$$

for $\lambda = 1/2$, $\nu = 2k+3/2$, k a sufficiently large positive integer.

3. **Preliminaries.** In verifying (2) for appropriate λ and ν, some preliminary results will be needed. The first is a corrected version of Wilf's formula (8), which we establish in a somewhat extended form:

$$\lim_{\nu\to\infty} \frac{\nu^\lambda \int_0^{j_{\nu 1}} t^{-\lambda} J_\nu(t)\,dt}{\nu^\lambda \int_0^\infty t^{-\lambda} J_\nu(t)\,dt} = \frac{1}{3} + \frac{1}{3}\int_0^c [J_{1/3}(t) + J_{-1/3}(t)]\,dt \tag{3}$$

$$\doteq 1.2743521,$$

where $\lambda > -1/2$, $j_{\nu 1}$ is the first positive zero of $J_\nu(t)$, and c is the least positive zero of $J_{1/3}(t) + J_{-1/3}(t)$.

PROOF OF (3). The denominator of the first member of (3) is equal to $(\nu/2)^\lambda \Gamma[(\nu+1-\lambda)/2]/\Gamma[(\nu+1+\lambda)/2]$ (cf., e.g., [**4**, p. 414 (11)]),

Received by the editors February 23, 1965.

[1] This note was written while the first-named author was enjoying the hospitality of the Mathematics Institute, Aarhus University, Denmark.

and so has limit equal to 1. The limit of the numerator is equal to the subsequent members of (3) by [4, p. 409 (4)].[2]

Another result to be used is

$$(4)^3 \qquad \int_0^{j_{\nu 2}} t^{-\lambda} J_\nu(t)\,dt > 0 \qquad \begin{cases} \text{(i)} & \lambda > 0, \quad \nu > -1 \quad \text{or} \\ \text{(ii)}^4 & \lambda > -1/2, \quad \nu > 1/2, \end{cases}$$

with $\lambda < \nu + 1$ (to insure convergence of the integral at the origin) where $j_{\nu 2}$ is the second positive zero of $J_\nu(t)$.

Proof of (4)(i). For small positive ϵ, the second mean-value theorem applies so that

$$\int_0^{j_{\nu 2}} t^{-\lambda} J_\nu(t)\,dt > \int_\epsilon^{j_{\nu 2}} t^{-\lambda} J_\nu(t)\,dt,$$

$$= \epsilon^{-\lambda} \int_\epsilon^{\eta} J_\nu(t)\,dt, \qquad \epsilon < \eta < j_{\nu 2}.$$

If $\eta \leqq j_{\nu 1}$, then this last integral is positive for $\nu > -1$, and (4)(i) is proved. If $\eta > j_{\nu 1}$, then this last integral clearly exceeds

$$\epsilon^{-\lambda} \int_\epsilon^{j_{\nu 2}} J_\nu(t)\,dt$$

and this, in turn, is positive for sufficiently small $\epsilon > 0$, in view of R. G. Cooke's result [3] that

$$\int_0^{j_{\nu 2}} J_\nu(t)\,dt > 0, \qquad \nu > -1.$$

Proof of (4)(ii). The same argument applies here to

$$\int_0^{j_{\nu 2}} t^{-(\lambda+1/2)} [t^{1/2} J_\nu(t)]\,dt, \qquad \lambda > -1/2, \quad \nu > 1/2,$$

in view of E. Makai's result [6] that

$$\int_0^{j_{\nu 2}} t^{1/2} J_\nu(t)\,dt > 0, \qquad \nu > 1/2.$$

---

[2] The results of [4] are summarized and extended in [5].

[3] Z. Ciesielski has mentioned that (4) (i) and (ii) can be inferred from the Cooke and Makai results, respectively, also via Theorem 1a of [1].

[4] Here the Bessel function of the first kind, $J_\nu(t)$, can be replaced by an *arbitrary* solution of the Bessel equation, say $\mathfrak{C}_\nu(t)$, normalized so as to be positive for t between zero and the first positive zero, with the parameters λ, ν restricted so as to insure convergence of the integral. This extension arises because [6], used in the proof of (4) (ii), covers this case.

A simplified version (cf. [2]) of these proofs (the introduction of ϵ being superfluous) shows that

$$(5)\qquad (-1)^p \int_{j_{\nu p}}^{j_{\nu,p+2}} t^{-\lambda} J_\nu(t)dt > 0 \begin{cases} \text{(i)} & \lambda \geqq 0, \quad \nu > -1 \quad \text{or} \\ \text{(ii)} & \lambda > -1/2, \quad \nu > 1/2, \end{cases}$$

where $j_{\nu p}$ is the pth positive zero of $J_\nu(t)$, $p=1, 2, \cdots$.

4. **Proof of the inequality.** That $h_{\nu\lambda}(\theta)<1$ for $\lambda<\nu+1$, and *either* $\lambda \geqq 0$, $\nu>-1$ *or* $\lambda>-1/2$, $\nu>1/2$ follows at once by combining (4) and (5), since they imply the positivity of the numerator in (1), for all $\theta \neq 0$. The denominator is also positive (its value is contained in the proof of (3)).

To show that $h_{\nu\lambda}(\theta)>-1$ for appropriate ν, λ, it suffices to show, as Wilf points out [7(a), p. 937], that the ratio of the integrals in (1) is less than 2. From (4) and (5) it is clear that the maximum of this ratio is achieved for $\theta=j_{\nu 1}$. But, for $\lambda>-1/2$, and all sufficiently large ν, this ratio must be less than 2, since the constant term in (3) is $1.2743521<2$.

Thus, (2) *is established for all sufficiently large* ν, *if* $\lambda>-1/2$ *and* $\lambda<\nu+1$. *In particular,* (2) *holds for* $\lambda=1/2$, $\nu=2k+3/2$, *for all sufficiently large positive integers* k, the case relevant to [7(a)].

References

1. Z. Ciesielski, *A note on some inequalites of Jensen's type*, Ann. Polon. Math. **4** (1958), 269–274.

2. R. G. Cooke, *Gibbs' phenomenon in Fourier-Bessel series and integrals*, Proc. London Math. Soc. **27** (1928), 171–192.

3. ———, *A monotonic property of Bessel functions*, J. London Math. Soc. **12** (1937), 180–185.

4. Lee Lorch and Peter Szego, *A singular integral whose kernel involves a Bessel function*, Duke Math. J. **22** (1955), 407–418; Corrections and a remark, ibid., **24** (1957), 683.

5. ———, *A singular integral whose kernel involves a Bessel function*. II, Acta Math. Acad. Sci. Hungar. **13** (1962), 203–217.

6. E. Makai, *On a monotonic property of certain Sturm-Liouville functions*, Acta Math. Acad. Sci. Hungar. **3** (1952), 165–172.

7. (a) H. S. Wilf, *The stability of smoothing by least squares*, Proc. Amer. Math. Soc. **15** (1964), 933–937.

(b) Errata, Proc. Amer. Math. Soc. **17** (1966), p. 542.

University of Alberta, Edmonton and
Ampex Corporation