A THEOREM ON LOCAL ISOMETRIES

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A mapping ϕ of a G-space R (Busemann [**1**]) on itself is a *locally isometric* mapping if for each $x \in R$ there is a number $\eta_x > 0$ such that ϕ maps the spherical neighborhood $S(x, \eta_x)$ isometrically on $S(\phi(x), \eta_x)$. The problem we are concerned with is that of determining conditions on a G-space R under which every locally isometric mapping of R on itself is an isometry. Several such conditions have recently been given by Busemann [**1**, §27], [**2**], Szenthe [**4**], [**5**], and the author [**3**]. In this paper we are concerned with the more general of the conditions given by Szenthe [**5**].

For a fixed point $p \in R$, consider the collection $G(p)$ of all geodesic curves which begin and end at p, and which do not contain subarcs traversed more than once. For $h \in G(p)$, let $l(h)$ denote the length of h. Let $\lambda_i(p)$ and $\lambda_s(p)$ equal, respectively, inf $l(h)$ and sup $l(h)$ for all $h \in G(p)$. Put $\lambda_i(p) = \infty$ and $\lambda_s(p) = 0$ if $G(p)$ is empty. Let

$$\lambda_i = \inf_{p \in R} \lambda_i(p); \qquad \lambda_s = \sup_{p \in R} \lambda_s(p).$$

Szenthe has proved [**5**] that if $\lambda_i > 0$ and $\lambda_s < \infty$, then every locally isometric mapping of R on itself is an isometry.

Szenthe's condition provides a solution to a problem suggested by Busemann [**1**, p. 405] in that it applies to a cylinder with euclidean metric. It is known [**1**] that every locally isometric mapping of a compact G-space on itself is an isometry. Szenthe's condition, however, fails to hold in every compact space since, as he points out [**5**, p. 441], $\lambda_s = \infty$ on a torus with euclidean metric. Our purpose here is to present a condition more general than Szenthe's which holds in every compact space.

Let ϕ denote a locally isometric mapping of a G-space R on itself. Let $\rho(p)$ be the supremum of those numbers ρ such that if x, y are in the spherical neighborhood $S(p, \rho)$, $x \neq y$, then there exists a point $z \neq y$ such that $xy + yz = xz$. We shall make use of the following properties of ϕ which are found in Busemann [**1**, §27].

(1) *If ϕ is* 1-1 *then ϕ is an isometry.*

(2) *If $\phi(p_1) = \phi(p_2) = p$, $p_1 \neq p_2$, then $p_1p_2 \geqq 2\rho(p)$.*

(3) *The number of points of R which lie over a given point of R is countable, and is the same for different points of R.*

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The following fact is an easy consequence of propositions (27.4) and (27.11) of Busemann [1].

(4) *If* $a, b \in R$ *are given, and if* $\phi(\bar{a}) = a$, *then there is a point* $\bar{b} \in R$ *such that* $\phi(\bar{b}) = b$ *and* $\bar{a}\bar{b} = ab$.

DEFINITION. A geodesic curve $h \in G(p)$ is called a *circular loop* at p provided there exists a z such that

$$h = T_1(p, z) \cup T_2(p, z)$$

where $T_1(p, z)$ and $T_2(p,z)$ denote (necessarily) distinct metric segments with endpoints p and z.

Let $Q(p)$ denote the collection of all circular loops at p. For $h \in Q(p)$, let $l(h)$ denote the length of h. Further, let

$$l_i(p) = \inf[l(h) \colon h \in Q(p)],$$
$$l_s(p) = \sup[l(h) \colon h \in Q(p)],$$

If $Q(p) = \varnothing$, take $l_i(p) = \infty$ and $l_s(p) = 0$. Then let

$$l_i = \inf[l_i(p) \colon p \in R],$$
$$l_s = \sup[l_s(p) \colon p \in R].$$

It will be helpful to establish the following lemma before turning to the theorem.

LEMMA. *Let* ϕ *be a locally isometric mapping of* R *on itself, and suppose that* $\phi(p_1) = p$. *Let*

$$W = \{x \in R \colon \phi(x) = p,\ x \neq p_1\}.$$

If $W \neq \varnothing$ *then there is at least one point* $p_2 \in W$ *such that* $p_1p_2 = \inf\ [p_1x \colon x \in W]$. *Further, if* $T(p_1, p_2)$ *is a segment joining* p_1 *and* p_2, *then* $\phi(T(p_1, p_2)) \in Q(p)$.

PROOF. The first part of the conclusion follows easily. If $x, y \in W$, $x \neq y$, then $xy \geqq 2\rho(p)$ by (2). Hence by finite compactness of R, the set of points $x \in W$ such that p_1x is less than a given number is finite.

We now prove that $\phi(T(p_1, p_2)) \in Q(p)$. Let z_1 denote the midpoint of $T(p_1, p_2)$. By (4), there is a point p_3 such that $\phi(p_3) = p$ and $z_1p_3 = zp$, where $z = \phi(z_1)$. If $p_3 = p_1$ then $pz = p_1z_1 = (1/2)p_1p_2$. Otherwise

$$p_1p_2 = p_1z_1 + z_1p_2 \geqq p_1z_1 + zp = p_1z_1 + z_1p_3 \geqq p_1p_3 \geqq p_1p_2,$$

and thus in either case $pz = p_2z_1 = (1/2)p_1p_2$.

Let $T(p_1, z_1)$ and $T(z_1, p_2)$ be subsegments of $T(p_1, p_2)$. Then $\phi(T(p_1, z_1))$ and $\phi(T(z_1, p_2))$ are segments since each is a geodesic arc whose length is equal to the distance between its endpoints. Therefore,

$$\phi(T(p_1, p_2)) = \phi(T(p_1, z_1)) \cup \phi(T(z_1, p_2)) \in Q(p).$$

THEOREM. *If R is a G-space for which $l_i > 0$ and $l_s < \infty$, then every locally isometric mapping of R on itself is an isometry.*

PROOF. Let ϕ be a locally isometric mapping of R on itself and suppose that ϕ is not an isometry. Let $p \in R$. We define the sequence $\{p_n\}$ as follows. By (1) ϕ is not 1-1, so by the lemma, for each positive integer n there is a point $\bar{p}_n$ such that $\phi(\bar{p}_n) = \phi^{n+1}(p)$ and

$$\bar{p}_n \phi^n(p) = \inf[x\phi^n(p) : \phi(x) = \phi^{n+1}(p),\ x \neq \phi^n(p)].$$

Using (4) and the fact that ϕ^n is also a local isometry, let p_n be a point such that $\phi^n(p_n) = \bar{p}_n$ and $pp_n = \phi^n(p)\bar{p}_n$.

By the lemma ϕ maps $T(\phi^n(p), \bar{p}_n)$ into an element of $Q(\phi^{n+1}(p))$, which has length $\phi^n(p)\bar{p}_n$, and hence $pp_n \leq l_s$. Further, $p_m \neq p_n$ if $n \neq m$ since, assuming $n < m$, $\phi^m(p_m) = \bar{p}_m$ while $\phi^m(p_n) = \phi^m(p) \neq \bar{p}_m$. Therefore, since R is finitely compact, there are integers k, l such that $p_k p_l < l_i$ where $k \neq l$. Assume $k < l$. Then $\phi^l(p_l) = \bar{p}_l$ and $\phi^l(p_k) = \phi^l(p)$. Thus $\bar{p}_l \phi^l(p) \leq p_l p_k < l_i$. Since $T(\phi^l(p), \bar{p}_l)$ is mapped by ϕ into an element of $Q(\phi^{l+1}(p))$ which has length $\bar{p}_l\phi^l(p)$, we have a contradiction.

That our theorem is more general than Szenthe's is evident; the collection $Q(p)$ is contained in the collection $G(p)$. It is easily seen that our condition also holds in all compact spaces. An element of $Q(p)$ is composed of two distinct segments $T_1(p, z)$ and $T_2(p, z)$. In a compact space the lengths of such segments are bounded above by the diameter of the space and bounded below by inf $[\rho(p) : p \in R]$ which is positive (cf. [**1**, p. 39]).

REFERENCES

1. H. Busemann, *The geometry of geodesics*, Academic Press, New York, 1955.

2. ———, *Geometries in which the planes minimize area*, Ann. Mat. Pura Appl. **55** (1961), 171–189.

3. W. A. Kirk, *On locally isometric mappings of a G-space on itself*, Proc. Amer. Math. Soc. **15** (1964), 584–586.

4. J. Szenthe, *Über lokalisometrische Abbildungen von G-Räumen auf sich*, Ann. Mat. Pura Appl. **55** (1961), 37–46.

5. ———, *Über metrische Räume, deren lokalisometrische Abbildungen Isometrien sind*, Acta Math. Acad. Sci. Hungar. **13** (1962), 433–441.

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