ON FUNCTIONS THAT COMMUTE WITH FULL FUNCTIONS

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A continuous function f mapping the unit interval $[0, 1]=I$ onto itself is said to be *full* if I can be partitioned into a finite number of subintervals J_i such that f maps each J_i homeomorphically onto I. The number of subintervals J_i will be called the *number of branches of* f. H. Cohen [2] showed that two full functions which commute (under composition) must have a common fixed point. In [1] Baxter and Joichi investigated the question of which continuous functions commute with full functions. The author conducted a similar investigation independently, and in this note we give some extensions of the results in [1]. Henceforth, it will be assumed that all functions considered are continuous.

Following Baxter and Joichi we define a *hat function* to be a piecewise linear full function whose derivative has constant absolute value. A full function f will be called *regular* if there is a homeomorphism ϕ of I onto I such that $\hat{f}=\phi f\phi^{-1}$ is a hat function. Baxter and Joichi show that if g commutes with a hat function f with at least two branches, then either g is a hat function or g is constantly equal to a fixed point of f. From this it follows that if g commutes with a regular full function f having two or more branches then g is either full and regular or it is constantly equal to a fixed point of f.

An example is given in [1] of a full function with two branches which commutes with a nonconstant, nonfull function. This shows that the above result does not hold for irregular full functions. However, the following generalization is valid.

THEOREM 1. *Let f be a full function with $n\geqq 2$ branches. There is a continuous monotone increasing function ϕ mapping I onto I and a hat function $\bar{f}$ with n branches such that $\phi f=\bar{f}\phi$. If g commutes with f, there is a continuous function $\bar{g}$ such that $\phi g=\bar{g}\phi$ and $\bar{g}$ commutes with $\bar{f}$. Furthermore, ϕ is a homeomorphism if and only if f is regular.*

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Combining the results of Baxter and Joichi with Cohen's theorem, we see that if g commutes with a regular full function f, then f and g have a common fixed point. Using Theorem 1 we obtain

THEOREM 2. *Let f be a full function. If g commutes with f, then f and g have a common fixed point.*

PROOF OF THEOREM 1. For each positive integer k, let

$$0 = t(k, 0) < t(k, 1) < \cdots < t(k, n^k) = 1$$

be the points where f^k, the kth iterate of f, assumes the values 0 and 1. Define ϕ by

$$\phi(x) = \sup\left\{\frac{i}{n^k} \mid k > 0,\ 0 \leqq i \leqq n^k,\ t(k, i) \leqq x\right\}.$$

Clearly ϕ is monotone increasing, $\phi(0)=0$ and $\phi(1)=1$. Since $t(k, i) = t(k+1, ni)$, $\phi(t(k, i)) = i/n^k$. Hence, $\phi(I)$ is dense in I. But ϕ is monotone so this implies that ϕ is continuous.

Let

$$T^k = \{t(k, i) \mid 0 \leqq i \leqq n^k\}$$

and let $T = \bigcup_k T^k$. Then T is the set of points in I which are mapped into $\{0, 1\}$ by some iterate of f. Now f is onto and $f(\{0, 1\}) \subset \{0, 1\}$ so $f(T) = f^{-1}(T) = T$. Furthermore, if $0 \leqq x \leqq y \leqq 1$, then a necessary and sufficient condition for $\phi(x)$ to be equal to $\phi(y)$ is that the interval $[x, y]$ contain at most one point from the set T.

Define $\bar{f}$ by

$$\bar{f}(x) = \phi f(\phi^{-1}(x)).$$

To show that $\bar{f}$ is well defined we must show that $\phi(x) = \phi(y)$ implies that $\phi f(x) = \phi f(y)$. Suppose not. Then there are $x, y \in I$ such that at most one point of T is between x and y and two points of T are between $f(x)$ and $f(y)$. But these two points are the images under f of a pair of points between x and y. This is a contradiction since $f^{-1}(T) = T$.

We clearly have $\bar{f}\phi = \phi f$. Let Q be a closed subset of I. Then $\bar{f}^{-1}(Q) = \phi(f^{-1}\phi^{-1}(Q))$. Now $f^{-1}\phi^{-1}(Q)$ is closed and hence compact by the continuity of f and ϕ. Therefore, $\phi(f^{-1}\phi^{-1}(Q))$ is compact and hence closed so $\bar{f}$ is continuous.

To show that $\bar{f}$ is a hat function with n branches, it is sufficient to show that, for each $k > 0$ and each i, $0 \leqq i < n$, $\bar{f}$ maps the points $i/n + j/n^{k+1}$, $0 \leqq j \leqq n^k$, monotonically onto the points l/n^k, $0 \leqq l \leqq n^k$.

This follows from

$$\bar{f}\left(\frac{i}{n}+\frac{j}{n^{k+1}}\right)$$
$$=\bar{f}\phi(t(k+1, n^k i+j)) = \phi f(t(k+1, n^k i+j))$$
$$=\begin{cases}\phi(t(k,j)) = \dfrac{j}{n^k} \text{ if } f \text{ is increasing on } [t(1, i), t(1, i+1)],\\[2ex] \phi(t(k, n^k - j)) = \dfrac{n^k - j}{n^k} \text{ if } f \text{ is decreasing on } [t(1, i), t(1, i+1)].\end{cases}$$

Now suppose that g commutes with f. By the above argument, to show the existence of a continuous $\bar{g}$ satisfying $\phi g=\bar{g}\phi$ it suffices to show that $\phi(x)=\phi(y)$ implies $\phi g(x)=\phi g(y)$.

Suppose not. Then for some $x, y\in I$ with $x<y$, $\phi(x)=\phi(y)$ and $\phi g(x)\neq\phi g(y)$. If there is a point $t\in T$ with $x<t<y$, it is the only element of T in $[x, y]$. Now $\phi(x)=\phi(t)=\phi(y)$ but either $\phi g(x)\neq\phi g(t)$ or $\phi g(t)\neq\phi g(y)$. Replacing $[x, y]$ by either $[x, t]$ or $[t, y]$, we may assume that there is no element $t\in T$ with $x<t<y$. Hence f^k is monotone on $[x, y]$ for every k.

Since $\phi g(x)\neq\phi g(y)$, for k sufficiently large there are an arbitrarily large number of consecutive points from T^k between $g(x)$ and $g(y)$. Hence, for any r there is a k and a monotone sequence $x_1, \cdots, x_r$ of points between $g(x)$ and $g(y)$ such that f^k alternately assumes the values 0 and 1 on this sequence. The sequence $x_1, \cdots, x_r$ is the image under g of a monotone sequence $y_1, \cdots, y_r$ of points in $[x, y]$. To see this, let $a, b\in I$. Now $g([a, b])$ is a closed interval containing $g(a)$ and $g(b)$. Hence, if c is between $g(a)$ and $g(b)$, then c is the image under g of a point between a and b. Since x_1 and x_r are between $g(x)$ and $g(y)$, there are points y_1 and y_r between x and y with $g(y_1)=x_1$ and $g(y_r)=x_r$. Now x_2 is between $x_1=g(y_1)$ and $x_r=g(y_r)$, so there is a point y_2 between y_1 and y_r with $g(y_2)=x_2$. Continuing in this fashion, we construct a sequence of points $y_1, \cdots, y_r$ in $[x, y]$ with $g(y_i)=x_i$ and y_i between y_{i-1} and y_r for $1<i<r$. This is the required monotone sequence.

The function $f^kg=gf^k$ alternately assumes the values 0 and 1 on the sequence $y_1, \cdots, y_r$. Since f^k is monotone on $[x, y]$, the sequence $f^k(y_1), \cdots, f^k(y_r)$ is monotone and g alternately assumes the values 0 and 1 on this sequence. But r may be arbitrarily large, so this contradicts the continuity of g.

Since ϕ is onto, to show that $\bar{f}\bar{g}=\bar{g}\bar{f}$ it suffices to show that $\bar{f}\bar{g}\phi=\bar{g}\bar{f}\phi$. This follows from the relations $\bar{f}\phi=\phi f$, $\bar{g}\phi=\phi g$ and $fg=gf$.

To see the final remark, observe that if ϕ is a homeomorphism then

$\phi f \phi^{-1} = \bar{f}$ so f is regular. Conversely, if f is regular then the set T is dense in $[0, 1]$. This implies that ϕ is 1-1 and hence a homeomorphism.

Before proceeding to the proof of Theorem 2, we need a

LEMMA. *Let f and g be continuous functions from $[a, b]$ to $[a, b]$ which commute. If f is monotone then f and g have a common fixed point.*

PROOF. If f is decreasing it has a unique fixed point x_0. Now $fg(x_0) = gf(x_0) = g(x_0)$ so $g(x_0) = x_0$. If f is increasing, let x_0 be a fixed point of g. The sequence $\{x_n\}$ defined by $x_n = f(x_{n-1})$ for $n > 0$ is monotone, so it has a limit which is a common fixed point of f and g.

PROOF OF THEOREM 2. If f has only one branch, then f is monotone and the conclusion follows from the lemma. If f has more than one branch then the hypotheses of Theorem 1 are satisfied. By [1] and [2], $\bar{f}$ and $\bar{g}$ have a common fixed point x_0. Let $[a, b] = \phi^{-1}(x_0)$. Then $\phi f([a,b]) = \bar{f}\phi([a,b]) = \bar{f}(x_0) = x_0$, so $f([a,b]) \subset [a,b]$. Similarly, $g([a, b]) \subset [a, b]$.

Suppose f is not monotone on $[a, b]$. Then there is a $t \in T^1$ with $a < t < b$. Now $f(t) \in [a, b]$ and $f(t) \in T$, but $[a, b]$ contains at most one point from T, so $f(t) = t$. However, $f(t)$ is either 0 or 1 and neither of these points are in the interior of $[a, b]$. Hence, f is monotone on $[a, b]$. By the lemma, f and g have a common fixed point in $[a, b]$.

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