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AN ADDITION TO ADO'S THEOREM[1]

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The main purpose of this note is to point out the following strengthened (with respect to the nilpotency property) form of the theorem on the existence of a faithful finite-dimensional representation of a finite-dimensional Lie algebra.

THEOREM 1. *Let L be a finite-dimensional Lie algebra over an arbitrary field, and let α denote the adjoint representation of L. There exists a faithful finite-dimensional representation ρ of L such that $\rho(x)$ is nilpotent for every element x of L for which $\alpha(x)$ is nilpotent.*

For the suggestion that this nilpotency property of ρ might be secured I am indebted to Leonard Ross who used the characteristic 0 case of Theorem 1 in his proof of Ado's Theorem for graded Lie algebras (Thesis, *Cohomology of graded lie algebras*, University of California, Berkeley, 1964).

In the case of characteristic 0, it is known that there exists a faithful finite-dimensional representation of L whose restriction to the maximum nilpotent ideal of L is nilpotent [**1**, pp. 202–203]. Hence, in order to establish Theorem 1 in the case of characteristic 0, it suffices to make the following observation:

Let L be a finite-dimensional Lie algebra over a field of characteristic 0, *and let M be a finite-dimensional L-module on which the maximum nilpotent ideal N of L is nilpotent. Let x be an element of L whose adjoint image $\alpha(x)$ is nilpotent. Then x is nilpotent on M.*

PROOF. Write $L=S+R$, where R is the radical of L and S is a semisimple subalgebra of L. Accordingly, write $x=s+r$, with s in S and r in R. Since $\alpha(x)$ is nilpotent, it is clear that the adjoint representation of S sends s onto a *nilpotent* derivation of S. Since S

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is semisimple, it follows from (the easy part of) [**1**, Chapter III, Theorem 17] that there is an element t in S such that $[s, t]=s$. Now regard L as a module for the *solvable* Lie algebra spanned by s and t, via the adjoint representation. By Lie's Theorem, the commutator subalgebra of this solvable Lie algebra is nilpotent on L, which means that $\alpha(s)$ is nilpotent. Since N and s are nilpotent on L and $[s, N]\subset N$, it follows by a familiar elementary argument that the Lie algebra spanned by s and N is still nilpotent on L. Since $[x, s]=[r, s]\in N$, we may apply the same argument again to conclude that the Lie algebra spanned by x, s and N is nilpotent on L. Hence r is nilpotent on L. Since r belongs to the radical of L, this gives $r\in N$.

Our above argument for showing that $\alpha(s)$ is nilpotent applies also to show that s is nilpotent on M. Now s and N are nilpotent on M, and it follows as above that the Lie algebra spanned by s and N is nilpotent on M. Since this Lie algebra contains $s+r=x$, this proves that x is nilpotent on M.

In order to prove Theorem 1 in the case of nonzero characteristic, we must establish the following result concerning the center of the universal enveloping algebra of L.

THEOREM 2. *Let L be a finite-dimensional Lie algebra over the field F of nonzero characteristic p, and let U denote the universal enveloping algebra of L. Let C denote the center of LU. Then $\bigcap_{n=1}^{\infty} UC^n=(0)$.*

PROOF. It is well known that U is left Noetherian [**1**, Chapter V, Theorem 6]. Thus the ideal UC has a finite ideal basis, whence it is clear that there is a finite subset $(c_1, \cdots, c_k)$ of C such that $UC=Uc_1+\cdots+Uc_k$. We may take this subset such that it contains a nontrivial p-polynomial $x^{p^e}+a_1x^{p^{e-1}}+\cdots+a_ex$, with each a_i in F, corresponding to each element x of a chosen basis of L; see [**1**, Chapter VI, Lemma 5]. Let R denote the subring of U that is generated by F and $(c_1, \cdots, c_k)$. Then R is a Noetherian commutative ring, and U is finitely generated as an R-module, because of the presence of the above p-polynomials among the c_i's (see [**1**, Chapter V, Lemma 4]).

Let J be the ideal of R that is generated by $(c_1, \cdots, c_k)$. Since U has no nonzero divisors of 0 and $J\subset LU$, it is clear that $(1+j)u\neq 0$ for every j in J and every nonzero u in U. Hence it follows from the well-known generalized form of Krull's Theorem [**2**, p. 253] that $\bigcap_{n=1}^{\infty} J^nU=(0)$. Now $UC^n=(UC)^n=(JU)^n=J^nU$, so that Theorem 2 is proved.

In particular, there exists an exponent n such that $(UC^n)\cap L=(0)$. The left multiplication action of L on U defines the structure of an

L-module on $U/(UC^n)$, and it is clear from the choice of n that this representation of L on $U/(UC^n)$ is faithful. Moreover, $U/(UC^n)$ is of finite dimension over F, because U is finitely generated as an R-module. Now let x be an element of L such that $\alpha(x)$ is nilpotent. Then there is an exponent e such that $\alpha(x)^{p^e}=0$, whence $x^{p^e}\in C$. Hence x is nilpotent on $U/(UC^n)$, so that the characteristic p case of Theorem 1 is proved.

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