PRIMITIVE IDEMPOTENTS IN GROUP ALGEBRAS

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1. **Introduction.** One of the reasons why character theory is important for the study of finite groups is that each complex representation is uniquely determined by its character. It is therefore natural to ask how one obtains the representation from the character. The main object of this paper is to give a method by which an irreducible representation can be constructed from a knowledge of the character of the representation and its behavior on various subgroups. The procedure is to construct a primitive idempotent which generates an irreducible left ideal (in the group algebra) affording the given representation. We do not know if this procedure can be applied to all finite groups. We do show that it works for all solvable groups and for certain classes of nonsolvable groups.

I wish to thank C. W. Curtis for bringing this problem to my attention and also for his assistance during the preparation of this paper. Thanks are due also to George Glauberman for providing the example in §2.

2. **Idempotents.** Let G be a finite group of order $|G|$ and let KG denote the group algebra of G over the complex field K. We shall consider a fixed irreducible left KG module, X with character χ. The element

$$e(\chi) = \chi(1)\,|\,G\,|^{-1} \sum_{g \in G} \chi(g)g^{-1}$$

is a central idempotent of KG [2, p. 236] with the property that $e(\chi)$ acts as the identity operator on X and annihilates any irreducible module not isomorphic to X.

The reader may consult Chapter VI of [2] for the properties of induced characters used below and for any unexplained notation.

THEOREM 1. *Suppose for some subgroup H of G, $\chi|H$ contains an irreducible character ν of H with multiplicity one. If ϵ is a primitive idempotent of KH such that $KH\epsilon$ affords ν, then $\epsilon e(\chi)$ is a primitive idempotent of KG such that $KG\epsilon e(\chi)$ affords χ.*

PROOF. We have $KH \subseteq KG$ so that $KG\epsilon$ is the module induced from

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$KH\epsilon$. Thus $KG\epsilon$ has character ν^G. Multiplication by $e(\chi)$ annihilates all the direct summands not isomorphic to X so $KG\epsilon e(\chi)$ is a direct sum of copies of X. The number of copies is $(\chi, \nu^G) = (\chi|H, \nu) = 1$. Hence $X \cong KG\epsilon e(\chi)$.

COROLLARY 1. *Suppose there is a chain of subgroups*

$$G = H_0 \supset H_1 \supset \cdots \supset H_r$$

and that χ_i is an irreducible character of H_i, $0 \leqq i \leqq r$, $\chi = \chi_0$, such that (a) *χ_{i+1} has multiplicity one in $\chi_i | H_{i+1}$, $0 \leqq i < r$ and* (b) $\chi_r(1) = 1$.

Let e_i denote the central idempotent $e(\chi_i)$ of KH_i. Then $e = e_0 e_1 \cdots e_r$ is a primitive idempotent of KG with $X \cong KGe$.

PROOF. The condition $\chi_r(1) = 1$ implies that the central idempotent $e(\chi_r)$ of KH_r is also a primitive idempotent of KH_r. The corollary now follows easily by induction from the theorem.

For the special case $r = 1$ and $\chi_1 = 1(H_1) =$ the 1-character of H_1, this corollary was known to Burnside [**1**, p. 306].

COROLLARY 2. *With the same assumptions as in Corollary* 1 *the representation affording χ can be written in the field E generated over the rational field by the values of the characters χ_i for $0 \leqq i \leqq r$.*

PROOF. The primitive idempotent e is an E-linear combination of the elements of G so KGe has a basis consisting of E-linear combinations of elements of G. Hence the matrices relative to this basis of the linear transformations induced on KGe by the elements of G have their entries in E.

This corollary gives an alternate proof of Brauer's splitting field theorem [**2**, p. 292] for those groups which satisfy the hypothesis of Corollary 1 for each irreducible character χ.

We now show that Corollary 1 actually applies to a large class of finite groups. We have the following first approximation.

THEOREM 2. *Let H be a normal subgroup of G with $[G:H]$ prime. Then any irreducible character of H contained in $\chi|H$ appears with multiplicity one.*

PROOF. By the proof of Theorem 3 of [**4**], $\chi|H$ is either irreducible or the sum of distinct conjugates of one of the irreducible characters of H appearing in $\chi|H$. In either case no character of H has multiplicity greater than one in $\chi|H$.

COROLLARY. *If G is a solvable group then the hypothesis of Corollary* 1 *holds for each irreducible character χ of G.*

PROOF. Since G is solvable we can choose the subgroups H_i so that

H_{i+1} is normal and of prime index in H_i, $0 \leqq i \leqq r$, $G = H_0$. By Theorem 2 there exists irreducible characters χ_i of H_i so that $\chi = \chi_0$ and χ_{i+1} has multiplicity one in $\chi_i | H_{i+1}$. If r is sufficiently large, H_r will be abelian so that $\chi_r(1) = 1$.

The case of Corollary 1, most practical for actual computation, occurs when there is a subgroup H possessing a character of degree 1 appearing in $\chi | H$ with multiplicity one. Unfortunately such a subgroup does not always exist. The following example communicated to the author by George Glauberman shows that it may be necessary to take $r \geqq 2$ in Corollary 1 even for solvable groups.

Let p be a prime with $p \geqq 7$ and $p \equiv 3 \bmod 4$. Let G be the group generated by elements x, y, z which satisfy

$$x^p = y^p = z^p = (x, z) = (y, z) = 1 \quad \text{and} \quad (x, y) = z,$$

where $(u, v) = u^{-1}v^{-1}uv$.

There is a subgroup A of the automorphism group of G such that A is isomorphic to the quaternion group of order 8. For the example take the split extension GA of A with kernel G. Let β be the irreducible character of A with degree two and let α be an irreducible character of GA with the property $\alpha | G$ is an irreducible character of G with degree p. Then $\alpha\beta$ is an irreducible character of GA such that no character of a subgroup with degree one has multiplicity one in $\alpha\beta$ on that subgroup.

3. **Remarks.** We would like to conclude by mentioning a few classes of groups, not all of whose members are solvable, but all of which satisfy the hypothesis of Corollary 1.

(a) The symmetric group S_n of permutations on n symbols. (See §28 of [2] for details.) Let χ be an irreducible character of S_n, D the young diagram corresponding to χ, and H_1 the subgroup of row permutations of D. Then the 1-character, $1(H_1)$, of H_1 has multiplicity one in χ.

(b) Let $G = SL(2, q) =$ group of 2×2 matrices with determinant 1 and entries from the field of q elements. Let H_1 be the subgroup of upper triangular matrices having only 1's on the main diagonal. A fairly long computation will show that if χ is an irreducible character of G, then $\chi | H_1$ contains a linear character of H_1 with multiplicity one.

(c) Let G be a Frobenius group—that is G has a proper normal subgroup N such that each nonidentity element of N commutes only with elements in N. Using results of W. Feit [**3**], M. Suzuki [**5**], J. Thompson [**6**], and part (b) above we can show that the hypothesis of Corollary 1 holds for any irreducible character of G.

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