DISCRETE COMPLEMENTED ALGEBRA

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I. Recently [5] the author introduced the concept of a discrete H^* -algebra. A two-sided H^* -algebra A is called discrete if each invariant closed ideal in A contains a two-sided projection (a two-sided projection is an idempotent which is both left and right self-adjoint). (An ideal I is said to be invariant if $I^r = \{x^r \mid x \in I\} = \{x^l \mid x \in I\} = I^l$.) It was shown that a simple proper discrete two-sided H^* -algebra is isomorphic to a canonical algebra. (A canonical algebra is an algebra of the type described in the example below with the additional assumption that the sequence $\{t(i)\}$ is bounded.)

The intent of this note is to obtain a similar result for complemented algebras. It is necessary to use a different technique since the involutions are not defined everywhere in the case of a general complemented algebra. In particular we define an invariant ideal in a different (but equivalent) fashion.

We continue to use terminology and notation of [3], [4], and [5]. In particular $l(R)(r(L))$ will denote the left (right) annihilator of a right (left) ideal $R(L)$ in an algebra; $R^p(L^p, l(R)^p, r(L)^p)$ will denote the orthogonal complement of $R(L, l(R), r(L))$.

II. Let A be a Banach algebra with a Hilbert space norm. A is called a complemented algebra [3] if the orthogonal complement of each right (left) ideal is again a right (left) ideal.

DEFINITION 1. A closed right (left) ideal $R(L)$ in a complemented algebra A is said to be invariant if $l(R^p) = l(R)^p$ ($r(L^p) = r(L)^p$) i.e. the orthogonal complement of the (appropriate) annihilator of an ideal coincides with the (appropriate) annihilator of its orthogonal complement.

From corollary to Lemma 3 in [5] it follows that in the case of a two-sided H^* -algebra this definition of an invariant ideal is equivalent to the definition given in [5]. Note also that an ideal I is invariant if and only if its orthogonal complement I^p is invariant.

It was established in [4] that in a semisimple complemented algebra A there are two involutions, $x \rightarrow x^r$ and $x \rightarrow x^l$, defined on dense subsets of A , such that $(yx, z) = (y, zx^r)$ and $(xy, z) = (y, x^lz)$ for all $y, z \in A$. If A is not semisimple we cannot speak about involutions since the right (left) adjoint $x^r(x^l)$ of a member x of A does not have to be unique. However, even in this case we may talk about a right

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(left) selfadjoint element: $x \in A$ is right (left) selfadjoint if $(yx, z) = (y, zx)$ ($(xy, z) = (y, xz)$ for all $y, z \in A$.) A right (left) selfadjoint (nonzero) idempotent is called a right (left) projection [3]. A two-sided projection is a (nonzero) idempotent which is both right and left selfadjoint [5, definition after Lemma 3].

DEFINITION 2. A complemented algebra A is said to be discrete if each invariant closed right ideal in A contains a two-sided projection.

EXAMPLE. Let J be an arbitrary set and let $t(i)$ be a real-valued function defined on J such that $1 \leq t(i)$ for each $i \in J$. Let M be the set of all complex-valued matrices $x = (x_{ij})$ defined on $J \times J$ such that $\sum_{i,j} t(i) |x_{ij}|^2 t(j)$ converges. Then M is a complemented algebra in the ordinary operations of matrices and the scalar product

$$(x, y) = \sum_{i,j} t(i) x_{ij} \bar{y}_{ij} t(j).$$

To see that M is a complemented algebra, note that M contains a dense subset of elements x having a right adjoint $x^r (x_{ij}^r = t(i)/t(j) \bar{x}_{ji})$ and a dense subset of elements y having a left adjoint $y^l (y_{ij}^l = t(j)/t(i) \bar{y}_{ji})$. Let us show that the algebra M is discrete. For each $i \in J$ let e^i be a member of M such that $e_{kj}^i = 1$ if $k=j=i$ and $e_{kj}^i = 0$ for every other pair k, j . Then $\{e^i\}_{i \in J}$ is a set of mutually orthogonal minimal two-sided projections such that $M = \sum_{i \in J} e^i M = \sum_{i \in J} M e^i$.

Let R be a closed invariant right ideal in M . Let $e \in \{e^i\}_{i \in J}$ be such that $eR \neq 0$. Let us write $e = a_1 + a_2 = b_1 + b_2$ with $a_1 \in R$, $a_2 \in R^p$, $b_1 \in l(R^p) = l(R)^p$, $b_2 \in l(R)$. Then $a_1 \neq 0$ (otherwise $eR = 0$). From $e = ee = (a_1 + a_2)e = a_1e + a_2e$ we conclude that $a_1 = a_1e$, $a_2 = a_2e$; similarly $b_1 = eb_1$, $b_2 = eb_2$. From Theorem 1 of [4] it follows that b_1, b_2 have left adjoints; for the same reason a_1, a_2 have right adjoints. In fact, as it can be easily seen, $a_1^r \in l(R)^p = l(R^p)$, $a_2^r \in l(R)$ (if $x \in l(R)$ then $(a_1^r, x) = (ea_1^r, x) = (e, xa_1) = 0$; if $x \in l(R^p) = l(R)^p$ then $(a_2^r, x) = (e, xa_2) = 0$). But $a_1^r + a_2^r = e^r = e = b_1 + b_2$ and so $a_1^r = b_1$, $a_2^r = b_2$. By symmetry we have $b_1^l = a_1$, $b_2^l = a_2$. Thus $b_1 = a_1^r = a_1^l$ and so $a_1b_1 = a_1a_1^r = a_1a_1^l$ is an element in R which is both right and left selfadjoint. Following the proof of Lemma 1 on p. 101 of [2] we now show that R contains a two-sided projection.

III. LEMMA 1. Let $\{e_\alpha\}$ be a sequence of mutually orthogonal two-sided projections in a proper complemented algebra A . (A complemented algebra A is proper if $r(A) = (0)$ [3, p. 50].) Then the ideals $R = \sum_\alpha e_\alpha A$ and $L = \sum_\alpha A e_\alpha$ are invariant.

PROOF. First note that $l(R) = \bigcap_\alpha l(e_\alpha)$ and $r(L) = \bigcap_\alpha r(e_\alpha)$. By direct verification we see that

(1) $R^p = r(L)$ and that

(2) $l(R) = L^p$

(for example $x \in R^p$ would imply $\|e_\alpha x\|^2 = (x, e_\alpha x) = 0$ for each α and $x \in r(L)$ would imply $(x, e_\alpha y) = (e_\alpha x, y) = 0$ for each α and each $y \in A$). From (1) we conclude that $L \subset l(R^p)$; from (2) one can see that $x \in L^p \cap l(R^p)$ would imply $xR = 0$ and $xR^p = 0$ which is only possible if $x = 0$ (note that $l(A) = 0$ also [3, p. 50]). Thus $l(R^p) = l(R)^p (= L)$. Similarly $r(L^p) = r(L)^p$.

In the next lemma and the theorem after it, we will use the term "simple" to mean "simple and semisimple" [4, p. 56]. (Note that in our terminology each simple algebra is proper [3, p. 50].)

LEMMA 2. *Each invariant right ideal R in a simple discrete complemented algebra A contains a minimal two-sided projection. The same is true about left ideals.*

PROOF. Let e be a two-sided projection in R . If we write e as a sum of mutually orthogonal primitive (minimal) left projections and apply Theorem 1 of [4] we can conclude that each element in eA has a left adjoint in Ae . Similarly each element in Ae has a right adjoint in eA . Thus $A_0 = eAe$ is a proper two-sided H^* -algebra. It is finite dimensional (e is the identity for A_0) and so by Theorem 3 of [5] A_0 contains a minimal two-sided projection. Lemma 2 now follows from the fact that $A_0 \subset R$.

THEOREM. *Each simple discrete complemented algebra A is isomorphic to an algebra M of the type described in the example above.*

PROOF. From the above lemmas it follows that A contains a (maximal) set $\{e_i\}_{i \in J}$ of mutually orthogonal minimal projections such that $A = \sum_i e_i A = \sum_i A e_i = \sum_{i,j} e_i A e_j$. Using this set we can proceed as in the proof of Theorem 4 in [5] (in the present case we do not have to prove that the function $t(i)$ is bounded above) (note also that each element in $e_i A e_j$ has both right and left adjoints).

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