EXTENSIONS OF COMPLETELY 0-SIMPLE SEMIGROUPS BY COMPLETELY 0-SIMPLE SEMIGROUPS

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Let S and T be disjoint semigroups, T having a zero element, O' . A semigroup $(V, \circ)$ is called an extension of S by T if it contains S as an ideal and if the Rees factor semigroup V/S is isomorphic to T . We shall say that V is determined by a partial homomorphism if there exists a partial homomorphism $A \rightarrow \bar{A}$ of $T \setminus O'$ into S such that $A \circ B = AB$ if $AB \neq O'$, $A \circ B = \bar{A}\bar{B}$ if $AB = O'$, $A \circ s = \bar{A}s$, $s \circ A = s\bar{A}$, and $s \circ t = st$ where s, t in S and the operations in S and T are denoted by juxtaposition. The purpose of this note is to give a necessary and sufficient condition that V be determined by a partial homomorphism when S is a completely 0-simple semigroup and T is a completely 0-simple semigroup. Since these partial homomorphisms are known mod group homomorphisms [1, p. 109, Theorem 3.14], our extensions may be given an explicit form. A corollary to our theorem will include an important theorem due to W. D. Munn [1, p. 143, Theorem 4.22]. Our result should have important applications to the study of finite semigroups and to semigroups with some finiteness condition.

If S is any subset of a semigroup, $\mathcal{E}(S)$ will denote the set of idempotents of S and S^* will denote the set of nonzero elements of S . $\mathcal{R}$, $\mathcal{L}$, $\mathcal{D}$ and $\mathcal{H}$ will denote Green's relations [1, p. 47]. If $a \in S$, R_a will denote the $\mathcal{R}$ -class containing a . If e and f are idempotents and $ef = fe = e$, we say e is under f and write $e < f$. Basic definitions are given in [1]. Likewise references to the fundamental work of Clifford, Green, and Munn will be found in [1].

LEMMA. *Let V be an extension of a completely 0-simple semigroup S by a completely 0-simple semigroup T , and let 0 be the zero of S (hence, 0 is also the zero of V). If there is some $E \in \mathcal{E}(T^*)$ such that $ESE = 0$, then V is given by the partial homomorphism $\xi: T^* \rightarrow S$ which maps every element of T^* to 0.*

PROOF. Let $E \in \mathcal{E}(T^*)$ such that $ESE = 0$. If $F \in \mathcal{E}(T^*)$, there exists $Y \in T^*$ such that $E \mathcal{R} Y$ and $Y \mathcal{L} F$ [1, p. 79, Theorem 2.51]. Thus,

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$EY = Y$ and there exists $U \in T^*$ such that $UY = F$. Let $s \in S$. Now, $ESY = ESEY = 0$, $EYsY = 0$, $YSY = 0$, $UYSY = 0$, $FSY = 0$, $FsUY = 0$, $FsF = 0$, and $FSF = 0$. Thus, $FSF = 0$ for all $F \in \mathcal{E}(T^*)$. Now suppose that $A \in R_E \cap L_F$ where $E, F \in \mathcal{E}(T^*)$. Let $h \in S$ and $hA \in L_g$ for some $g \in E_S$. Since $uhA = g$ for some $u \in S$, $gF = uhAF = g$. Thus, $Fg = FgF = 0$. Hence $gFg = g0 = 0 = gg = g$ and $hA = 0$. Similarly $Ah = 0$. Now define $A\xi = 0$ for all $A \in T^*$. Clearly ξ is a partial homomorphism of T^* into S . If $AB = s \in S$, $AB = (EA)B = Es = 0 = (A\xi)(B\xi)$.

THEOREM. *An extension V of a completely 0-simple semigroup S by a completely 0-simple semigroup T is given by a partial homomorphism if and only if under each nonzero idempotent of T there exists at most one nonzero idempotent of S .*

PROOF. Suppose that under each nonzero idempotent of T there exists at most one nonzero idempotent of S . By virtue of the lemma, we may assume that $ESE \neq 0$ for all $E \in \mathcal{E}(T^*)$. We will first show that under each nonzero idempotent $E \in T^*$, there exists a unique idempotent $e \in S^*$ and that $ESE = H_e U_0$. Let $a \in ESE$ and $a \neq 0$. There exists $x \in ESE$ such that $axa = a$. Thus $ax = e \in E(S^*)$ and $e < E$, i.e., e is the unique idempotent of S^* which lies under E . Hence $xa = e$ and $ESE \subseteq H_e U_0$. If $b \in H_e$, $bE = beE = be = b = Eb$ and $ESE = H_e U_0$. Let $A \in R_E \cap L_F$ where $E, F \in \mathcal{E}(T^*)$, and let e and f be the unique idempotents of S^* under E and F respectively. We will show that if $h \in S$, $hA \neq 0$ implies that $hA \in L_f$ and $Ah \neq 0$ implies that $Ah \in R_e$. We consider only the first case, the other case being similar. Now, $hA \in L_g$ for some $g \in \mathcal{E}(S^*)$. As in the proof of the lemma, we show that $gF = g$. There exists $k \in S^*$ such that $g \mathcal{L} k$ and $k \mathcal{R} f$. Thus, $kF = kgF = kg = k = fk = Ffk = Fk$ and $k \in FSF$. Hence $k \in H_f$ and $hA \in L_f$. Next suppose that A is also an element of $R_{E'} \cap L_{F'}$ where E' and $F' \in \mathcal{E}(T^*)$ and let e' and f' denote the unique idempotents under E' and F' respectively. We will show that $eAf = e'Af'$ and hence it will follow that we may write $A\xi = eAf$ where ξ is a single valued mapping of T^* into S . We first note that $FF' = F$, $F'F = F'$, $fF' = f$, and $f'F = f'$. Thus, $FSFF'SF' = FSFSF'$. Since $f \in SFS$, $SFS = S$, and $f = FfF' \in FSF F'SF' = H_f H_{f'} \cup 0$. Thus $H_f H_{f'} \neq 0$ and $ff' \in H_f H_{f'} = R_f \cap L_{f'} \subseteq S^*$ [1, p. 79, Theorem 2.52]. Since $ff' = Fff'F \in FSF$, $ff' \in H_f$, $ff'f = ff'$, $(ff')^2 = ff'$, and $ff' = f$. Similarly, $f'f = f'$, i.e. $f \mathcal{L} f'$. In an analogous manner, we show that $e \mathcal{R} e'$. If $e'E = 0$, $e'EE' = 0$ and $e' = 0$. Hence $e'E \neq 0$ and similarly $Ff' \neq 0$. Thus, $e'Af' = ee'EAFFf' = e(e'E)A(Ff')f = e(e'E)eAf(Ff')f = eAf$. We next show that ξ is a partial homomorphism of T^* into S . Let $B \in R_G \cap L_H$ where $G, H \in \mathcal{E}(T^*)$, and let g and h denote the unique idempotents of S^* under

G and H respectively. If $AB \neq O'$, $AB \in R_E \cap L_H$ [1, p. 79, Theorem 2.52]. Thus $(AB)\xi = eABh$ and $A\xi B\xi = eAfgBh = [(eA)f][g(Bh)]$. Since $A \mathcal{R} E$, $eA \mathcal{R} e$ and $eA \neq 0$. Similarly, $Bh \neq 0$ and hence $(AB)\xi = A\xi B\xi$. Now, let $b \in S$. If $eAfb \neq 0$, $eAfb = ((eA)f)b = (eA)b = e(Ab) = Ab$. If $Ab \neq 0$, we may reverse the steps. Thus $Ab = (A\xi)b$ in all cases. Similarly $bA = b(A\xi)$. Now suppose that $AB = s \in S$. Then, $s = sH = sh = (AB)h$. If $s \neq 0$, $A(Bh) = A\xi Bh = A\xi gBh = A\xi B\xi$ and $AB = A\xi B\xi \neq 0$. If $A\xi B\xi \neq 0$, $A\xi B\xi = ((eA)f)(g(Bh)) = eA(g(Bh)) = ((eA)g)(Bh) = (e(Ag))(Bh) = (Ag)(Bh) = A(g(Bh)) = A(Bh) = s$. Thus, in all cases, $AB = A\xi B\xi$.

Conversely, suppose that the extension V is given by a partial homomorphism ξ of T^* into S . First suppose that $A\xi = 0$ for some $A \in T^*$. Let $U = (A \in T^*: A\xi = 0)$ and let $U' = U \cup 0$. Clearly, U' is a nonzero ideal of T and hence $U' = T$, i.e., $A\xi = 0$ for all $A \in T^*$. In this case, if $E \in \mathcal{E}(T^*)$, 0 is the only idempotent under E . If $A\xi \neq 0$ for all $A \in T^*$, $E\xi$ is the unique idempotent of S^* under E .

COROLLARY. *An extension V of a completely simple semigroup S by a completely 0-simple semigroup T is given by a partial homomorphism if and only if under each nonzero idempotent of T there exists at most one idempotent of S .*

REMARK. In the statement of the theorem, we may replace T by a regular semigroup with zero in which every nonzero idempotent is primitive [1]. This follows since such a semigroup is an orthogonal sum of completely 0-simple semigroups [2].

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