FINITE INTERPOLATION FOR ANALYTIC FUNCTIONS WITH FINITE DIRICHLET INTEGRALS

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The finite interpolation problem for AD-functions on planar (zero genus) Riemann surfaces was completely solved by Sario [2] and Rodin [1]. We shall extend their result to the case of Riemann surfaces with finite genus.

THEOREM. *Let R be an open Riemann surface of finite genus. Given a finite number of distinct points ζ_k $(k=1, 2, \cdots, n)$ in R, local parameters z_k at ζ_k with $z_k(\zeta_k)=0$ $(k=1, 2, \cdots, n)$ and complex numbers $\alpha_{\nu k}$ $(\nu=0, 1, \cdots, m;\ k=1, 2, \cdots, n)$. Then there exists a bounded analytic function f with finite Dirichlet integral on R such that*

$$\text{(1)}\qquad \frac{d^\nu f}{dz_k^\nu}(\zeta_k) = \alpha_{\nu k} \qquad (\nu = 0, 1, \cdots, m;\ k = 1, 2, \cdots, n)$$

if and only if R does not belong to the class 0_{AD}.

PROOF. The necessity of the condition $R \notin 0_{AD}$ is evident. We have to show the solvability of (1) under the condition $R \notin 0_{AD}$. Since R has finite genus, $R \notin 0_{AD}$ implies the existence of a nonconstant ABD-function $F(z)$ on R. Let R^* be a closed Riemann surface which contains R as a subsurface. Choose a point ζ_0 in $R-\{\zeta_1, \zeta_2, \cdots, \zeta_n\}$ such that $F(\zeta_0) \neq F(\zeta_k)$ $(k = 1, 2, \cdots, n)$. For each fixed k $(k=1, 2, \cdots, n)$, by Riemann-Roch's theorem, there exists a meromorphic function $r_k(z)$ on R^* such that $r_k(z)$ has a simple pole at ζ_k and a pole of order n_k at ζ_0 and regular on $R^*-\{\zeta_0, \zeta_k\}$. Let m_k be the order of zero of the function $\prod_{j=1}^n (F(z)-F(\zeta_j))^{m+1}$ at ζ_k and let $s=\max\{m_k n_k;\ k=1, 2, \cdots, n\}$. Put

$$H(z) = (F(z) - F(\zeta_0))^s \prod_{j=1}^{n} (F(z) - F(\zeta_j))^{m+1},$$

which belongs to the class $ABD(R)$. By construction, $(d^{m_k}H/dz_k^{m_k})(\zeta_k) \neq 0$ and $(z_k r_k)(\zeta_k) \neq 0$. Hence for each ν $(\nu=0, 1, \cdots, m)$, we may set

$$H_{\nu k}(z) = \left[(\nu!)\cdot\frac{1}{m_k!}\cdot\frac{d^{m_k}H}{dz_k^{m_k}}(\zeta_k)\left((z_k r_k)(\zeta_k)\right)^{m_k-\nu}\right]^{-1}\cdot(r_k(z))^{m_k-\nu}\cdot H(z).$$

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Again from the construction it is easy to see that $H_{\nu k}$ belongs to $\mathrm{ABD}(R)$ and for each k $(k=1, 2, \cdots, n)$,

$$H_{0k}(\zeta_j) = \delta_{kj} \qquad (j = 1, 2, \cdots, n)$$

and moreover for each fixed ν $(\nu=1, 2, \cdots, m)$,

$$\frac{d^{\mu}H_{\nu k}}{dz_j^{\mu}}(\zeta_j) = 0 \qquad (\mu = 0, 1, \cdots, \nu-1; j = 1, 2, \cdots, n),$$

$$\frac{d^{\nu}H_{\nu k}}{dz_j^{\nu}}(\zeta_j) = \delta_{kj} \qquad (j = 1, 2, \cdots, n).$$

Define $m+1$ functions $P_\nu(z)$ $(\nu=0, \cdots, m)$ on R inductively by

$$P_\nu(z) = P_{\nu-1}(z) + \sum_{j=1}^{n}\left(\alpha_{\nu j} - \frac{d^{\nu}P_{\nu-1}}{dz_j^{\nu}}(\zeta_j)\right)H_{\nu j}(z) \qquad (\nu = 0, \cdots, m)$$

with $P_{-1}=0$. Then $f(z)=P_m(z)$ belongs to $\mathrm{ABD}(R)$ and satisfies (1).

COROLLARY. *Let R be an open Riemann surface of finite genus not belonging to the class 0_{AD} and $\mathfrak{F}=\mathfrak{F}((\zeta_k), (z_k), (\alpha_{\nu k}))$ be the class of all* AD-*functions f on R satisfying the interpolating condition* (1). *Then the class $\mathfrak{F}$ is not empty and there exists a unique function f_0 in $\mathfrak{F}$ such that*

$$D(f) = D(f_0) + D(f - f_0)$$

for any f in $\mathfrak{F}$ and a fortiori f_0 is the unique solution with minimum norm of the interpolation problem given by (1):

$$D(f_0) = \min\{D(f); f \in \mathfrak{F}\}.$$

PROOF. For each closed parametric disk K_k with local parameter z_k $(k=1, 2, \cdots, n)$ and for any relatively compact parametric disk U with local parameter z such that $\zeta_k\in U$ $(k=1, 2, \cdots, n)$, by the local subharmonicity of $|f'|^2$ for $f\in\mathfrak{F}$ and Cauchy's inequalities, we can find a constant c_U such that

$$(2) \quad |f_1(z) - f_2(z)|^2 + \sum_{k=1}^{n}\sum_{\nu=0}^{m}\left|\frac{d^{\nu}f_1}{dz_k^{\nu}}(z_k) - \frac{d^{\nu}f_2}{dz_k^{\nu}}(z_k)\right|^2 \leqq c_U D(f_1 - f_2)$$

for any $z\in U$, $z_k\in K_k$ $(k=1, 2, \cdots, n)$ and $f_1, f_2\in\mathfrak{F}$. Let $\{f_n\}$ be a sequence such that $\{f_n\}\subset\mathfrak{F}$ and $\lim_n D(f_n)=d=\inf\{D(f); f\in\mathfrak{F}\}$. Since $(f_n+f_{n+p})/2\in\mathfrak{F}$ and

$$\begin{aligned} D(f_n - f_{n+p}) &= 2(D(f_n) + D(f_{n+p})) - 4D\left(\frac{f_n + f_{n+p}}{2}\right) \\ &\leqq 2(D(f_n) + D(f_{n+p})) - 4d, \end{aligned}$$

we conclude that $\lim_n D(f_n - f_{n+p}) = 0$ for any p. This with (2) gives the existence of a function f_0 in $\mathfrak{F}$ such that $\lim_n D(f_n - f_0) = 0$ so that $D(f_0) = d$. For any $f \in \mathfrak{F}$ and any complex number λ, $f_0 + \lambda(f - f_0) \in \mathfrak{F}$. Hence $D(f_0 + \lambda(f - f_0)) \geqq D(f_0)$. Whence it follows that $D(f_0, f - f_0) = 0$. Therefore $D(f) = D(f_0 + (f - f_0)) = D(f_0) + D(f - f_0) = d + D(f - f_0)$. Thus $D(f) = d$ if and only if $f = f_0$.

References

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2. L. Sario, *Extremal problems and harmonic interpolation on open Riemann surfaces*, Trans. Amer. Math. Soc. **79** (1955), 362–377.

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