EXISTENCE OF POSITIVE HARMONIC FUNCTIONS[1]

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1. Consider an open Riemann surface R. In this note by a *distinguished subregion* G of R we understand that G is a subregion of R with nonempty analytic relative boundary ∂G and with noncompact closure $\bar{G}$. The purpose of this note is to give a simple proof of the following theorem of Parreau [**5**]:[2]

THEOREM. *For any distingished subregion G of an arbitrary open Riemann surface R, there exists a nonconstant positive harmonic function u on G with continuous boundary value zero on ∂G.*

It is interesting to compare the theorem[3] with the so-called "two domains criterion" due to Bader-Parreau [**1**] and Mori [**4**]: An open Riemann surface R does not belong to the class O_{HB} (resp. O_{HD}) if and only if there exist two disjoint distinguished subregions of R carrying nonconstant HB (resp. HD) functions with continuous boundary values zero on their relative boundaries. The theorem shows that the two domains criterion fails for the class O_{HP}.

Another consequence of the theorem is that the Martin compactification G^* of any distinguished subregion G, considered as a Riemann surface, always contains an HP-minimal point other than those HP-minimal points identified with points in ∂G, no matter whether $R \in O_G$ or not. For Martin's compactification and HP-minimal points see e.g. Constantinescu-Cornea [**2**].

The proof of the theorem will be given in §§2–4.

2. Let G be a distinguished subregion of an open Riemann surface R. Fix a point $p_0 \in G$. We denote by $g_G(p, q)$ the Green's function of G. We choose an arbitrary sequence $\{q_n\}$ of points in G converging to the ideal boundary of G, i.e. converging to the Alexandroff points of R. Following Martin [**3**] we set

$$u_n(p) = \frac{g_G(p, q_n)}{g_G(p_0, q_n)} \tag{1}$$

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[3] The thorem can be expressed simply as $G \notin SO_{HP}$. A comparison of this with the following is also interesting: $G \in SO_{HB}$ (resp. SO_{HD}) if $R \in O_G$, and $G \notin SO_{HB}$ (resp. SO_{HD}) if $R \notin O_G$ and ∂G is compact.

for p in $R-q_n$. Observe that $u_n(p_0)=1$. Since $u_n \in HP(\Omega)$ for any relatively compact subregion Ω and for sufficiently large n, $\{u_n\}$ constitutes a normal family. Therefore by chosing a suitable subsequence of $\{q_n\}$, we may assume that

$$(2) \qquad u_\infty(p) = \lim_{n\to\infty} u_n(p)$$

exists in G. Obviously $u_\infty \in HP(G)$ and $u_\infty > 0$ in G since $u_\infty(p_0) = \lim u_n(p_0) = 1$.

3. The proof will be complete if we show that u_∞ has the continuous boundary value zero on ∂G. Take an arbitrary open arc α in ∂G with compact closure. We only have to show that u_∞ has the continuous boundary value zero on α.

Join the two endpoints of α by a simple analytic arc γ in G so that the subregion F of G bounded by $\alpha \cup \gamma$ is simply connected. By the Riemann mapping theorem we can map F onto the open unit disk U by a conformal mapping ϕ. By Carathéodory's theorem ϕ can be assumed to be a topological mapping of $\overline{F}$ onto $\overline{U} = U \cup C$, where C denotes the unit circle. We set $\beta = \phi(\alpha)$, which is an open subarc of C, and

$$(3) \qquad v_k(z) = u_k(\phi^{-1}(z))$$

on U for $k=1, 2, \cdots, \infty$. Clearly $v_k \in HP(U)$ $(k=1, 2, \cdots, \infty)$ and in view of (2)

$$(4) \qquad v_\infty(z) = \lim_{n\to\infty} v_n(z)$$

on U. Moreover v_n is continuous on $\overline{U}$ for $n=1, 2, \cdots$ and by (1)

$$(5) \qquad v_n = 0 \qquad (n = 1, 2, \cdots)$$

on β. If we show that v_∞ has continuous boundary value zero on β, then the same conclusion follows for u_∞ on α.

4. Let μ_n be the regular Borel measure on C defined by

$$(6) \qquad d\mu_n(\zeta) = \frac{1}{2\pi} v_n(\zeta) \, |d\zeta|$$

for $\zeta \in C$ and $n=1, 2, \cdots$, where $|d\zeta|$ denotes the linear measure on C. By using the Poisson formula

$$(7) \qquad v_n(z) = \int_C \frac{1 - |z|^2}{|\zeta - z|^2} \, d\mu_n(\zeta)$$

for $z \in U$ and $n=1, 2, \cdots$. In particular $\mu_n(C)=v_n(0)$ and thus by (4) $\{\mu_n(C)\}$ is bounded. In view of the selection theorem (see e.g. Constantinescu-Cornea [2, p. 9]), by choosing a suitable subsequence of $\{\mu_n\}$, we may assume that there exists a regular Borel measure μ_∞ on C such that

$$\lim_{n\to\infty} \int_C \lambda(\zeta)\, d\mu_n(\zeta) = \int_C \lambda(\zeta)\, d\mu_\infty(\zeta) \tag{8}$$

for any real-valued continuous function λ on C. By (4), (7) and (8) with $\lambda(\zeta)=(1-|z|^2)\cdot|\zeta-z|^{-2}$ we obtain

$$v_\infty(z) = \int_C \frac{1-|z|^2}{|\zeta - z|^2}\, d\mu_\infty(\zeta) \tag{9}$$

in U. The definition (6) of μ_n with (5) shows that $\mu_n(\beta) = 0$ $(n=1, 2, \cdots)$. Therefore by (8) we conclude that

$$\mu_\infty(\beta) = 0. \tag{10}$$

Take a point z_1 in β and let ρ be the distance between z_1 and $C-\beta$. Then from (9) and (10) it follows that

$$0 < v_\infty(z) < 4\rho^{-2}\mu_\infty(C)(1 - |z|^2)$$

if $z \in U$ and $|z-z_1| < \rho/2$. This shows that v_∞ has continuous boundary value zero at z_1 and thus $v_\infty = 0$ on β.

References

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