NORMED LINEAR SPACES EQUIVALENT TO INNER PRODUCT SPACES

J. T. JOICHI[1]

1. **Introduction.** There are many conditions which are known to characterize those normed linear spaces (NLS) X which are inner product spaces (IPS), that is, conditions under which it is possible to define an inner product in X in such a way that it will induce the given norm (cf. [1]). However, little is known about when a NLS X is merely equivalent to an IPS, that is, when it is possible to define an inner product in X in such a way that the induced norm is equivalent to the given norm. Two norms $|\cdot|_1$ and $|\cdot|_2$ in X are called equivalent if there exists a constant $k \geqq 1$ so that $(1/k)|x|_1 \leqq |x|_2 \leqq k|x|_1$ for each x in X. The only such characterizations known to us are those in [3], [4], and [5]. In this paper, we give another such characterization. The statement and proof of our main theorem are in §3. In §2 we prove a preliminary result concerning the existence of invariant means on a certain space of bounded real-valued functions.

2. **Invariant means.** Let $(S, \geqq)$ be a semilattice, that is, a partially ordered set (reflexive, antisymmetric and transitive) in which every pair of elements have a least upper bound (write: $s \vee t = \text{l.u.b. } \{s, t\}$). Let $m(S)$ be the Banach space of bounded real-valued functions defined on S with the supremum norm. Since S is a directed set, each element f in $m(S)$ is a net and we let $c(S)$ be the closed linear subspace of $m(S)$ consisting of those functions which are convergent nets.

For f in $c(S)$, let $\phi(f) = \lim_S f$. Then ϕ is a bounded linear functional on $c(S)$. For f in $m(S)$, let

$$p(f) = \limsup_S f \equiv \lim_S \left[\sup \{f(t) : t \geqq s\}\right]$$

$$q(f) = \liminf_S f \equiv \lim_S \left[\inf \{f(t) : t \geqq s\}\right].$$

For each s in S, let L_s be the linear operator in $m(S)$ defined by,

$$(L_s f)(t) = f(s \vee t).$$

It is easily seen that,

(i) $\{L_s : s \text{ in } S\}$ is a commutative semigroup of bounded linear operators on $m(S)$,

(ii) $L_s(c(S)) \subseteq c(S)$, for each s in S,

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(iii) $\phi(L_s f)=\phi(f)$, for each s in S and f in $c(S)$, and

(iv) $p(L_s f)=p(f)$, for each s in S and f in $m(S)$.

Since, in addition, $\phi(f)=p(f)$ for each f in $c(S)$, by a generalization of the Hahn-Banach Theorem due to M. A. Woodbury [7] (cf. [6, p. 164]), there exists an extension Φ of ϕ to $m(S)$ (Φ in $m(S)^*$) such that

(a) $\Phi(f)\leqq p(f)$ for each f in $m(S)$, and

(b) $\Phi(L_s f)=\Phi(f)$ for each s in S and f in $m(S)$.

From (a) it follows that $q(f)=-p(-f)\leqq\Phi(f)$, that is,

(c) $q(f)\leqq\Phi(f)\leqq p(f)$ for each f in $m(S)$.

3. The main theorem.

THEOREM. *Let X be a NLS. Then a necessary and sufficient condition that X be equivalent to an IPS is that there exist a constant $k\geqq 1$ such that for each finite dimensional subspace M of X, there exists a linear mapping T_M of M into H (Hilbert space) such that $(1/k)|x| \leqq |T_M x| \leqq k|x|$ for each x in M.*

PROOF. The necessity is obvious. To prove the sufficiency, let S be the set of finite dimensional subspaces of X ordered by containment. Then S is a semilattice where $M\vee N=M+N$ (vector sum) for M and N in S. Thus, by the results of the previous section, there exists a functional Φ in $m(S)^*$ satisfying conditions (a), (b) and (c).

For x in X, define f_x in $m(S)$ by,

$$f_x(M) = \begin{cases} |T_M x| & \text{if } x \text{ is in } M, \\ 0 & \text{if } x \text{ is not in } M. \end{cases}$$

Now, let $n(x)=[\Phi(f_x^2)]^{1/2}$. We wish to show that n is an inner product norm in X equivalent to the original one.

Clearly, n is a nonnegative function and $n(\alpha x)=|\alpha|n(x)$ for any vector x and scalar α. Also, for any vector x, we have

$$n(x)^2 = \Phi(f_x^2)\begin{cases} \leqq p(f_x^2) = [\limsup_S f_x]^2 \leqq [k\,|x|\,]^2, \\ \geqq q(f_x^2) = [\liminf_S f_x]^2 \geqq [(1/k)\,|x|\,]^2, \end{cases}$$

so,

$$(1/k)\,|x| \leqq n(x) \leqq k\,|x|\,.$$

Thus, $n(x)=0$ if and only if $x=0$, and if n is a norm, it is equivalent to the original one. To show that n is a norm, it remains to show that n satisfies the triangle law. This will follow once we show that n satisfies the parallelogram law (and hence, is an inner product norm).

For vectors x and y in X, let N be the subspace spanned by x and y. Then, for any subspace M such that $M \supseteq N$, we have

$$\begin{aligned} 2[f_x^2(M) + f_y^2(M)] &= 2[|T_M x|^2 + |T_M y|^2] \\ &= |T_M x - T_M y|^2 + |T_M x + T_M y|^2 \\ &= f_{x-y}^2(M) + f_{x+y}^2(M) \end{aligned}$$

where the second equality follows since $T_M(M)$ is a subspace of Hilbert space and the norm in a Hilbert space does satisfy the parallelogram law. Applying Φ, it follows that

$$2[n(x)^2 + n(y)^2] = n(x-y)^2 + n(x+y)^2$$

that is, n satisfies the parallelogram law.

Now, let $B = \{x : n(x) \leqq 1\}$. Then n is the Minkowski functional of B and will satisfy the triangle law if B is convex (cf. [1, p. 11]). To see that B is convex, we note first that although the mapping $x \to f_x$ of X into $m(S)$ is not continuous, the mapping $x \to n(x)$ of X into the real numbers is continuous. Suppose now that x and y are vectors with $n(x) = n(y) = 1$ and for some α with $0 < \alpha < 1$, we have $n(\alpha x + (1-\alpha)y) > 1$. Then, by the continuity of n, there exists α_1 and α_2 so that $0 \leqq \alpha_1 < \alpha < \alpha_2 \leqq 1$, $n(\alpha_i x + (1-\alpha_i)y) = 1$ for $i = 1, 2$, and if $\alpha_1 < \beta < \alpha_2$ then $n(\beta x + (1-\beta)y) > 1$. Let $x_i = \alpha_i x + (1-\alpha_i)y$ for $i = 1, 2$. Then, by the parallelogram law, we have

$$\begin{aligned} 4 &= 2[n(x_1)^2 + n(x_2)^2] = n(x_1 + x_2)^2 + n(x_1 - x_2)^2 \\ &\geqq n(x_1 + x_2)^2 = 4n((1/2)(x_1 + x_2))^2 > 4. \end{aligned}$$

This contradiction establishes the convexity of B.

COROLLARY. *Let X be a NLS. Then a necessary and sufficient condition that X be equivalent to an IPS is that there exists a constant $k \geqq 1$ such that for each pair of finite dimensional subspaces M and N with* $\dim M = \dim N$, *there exists a linear mapping T of M onto N such that $(1/k)|x| \leqq |Tx| \leqq k|x|$ for each x in M.*

This corollary is an immediate consequence of our theorem and the following theorem due to Dvoretzky [2].

THEOREM (DVORETZKY). *Given ϵ $(0 < \epsilon < 1)$ and a positive integer m, there exists an integer $N = N(m, \epsilon)$ such that if C is any symmetric convex body in E^n (real Euclidean n-space), where $n \geqq N$, then there is a subspace E^m and a positive number r so that $B_{r(1-\epsilon)} \subseteq C \cap E^m \subseteq B_r$, where $B_s = \{x$ in E^m: $|x| \leqq s\}$.*

REMARK. We note that our theorem is applicable to any NLS whether or not it be separable. Also, the mappings T_M need only exist for some semilattice S of finite dimensional subspaces M of X such that the vector subspace $\bigcup\{M: M \text{ in } S\}$ is dense in X. In case X is separable, the mappings T_M need only exist for a sequence $\{M_n\}$ of finite dimensional subspaces such that $M_{n+1} \supseteq M_n$ and $\bigcup\{M_n: n=1, 2, \cdots\}$ is dense in X. In this latter case, the functional Φ is an ordinary Banach limit on the Banach space of bounded sequences.

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UNIVERSITY OF MINNESOTA