Using these estimates in (17), we see that

$$\log|F(z)| = \log|f(z)| + \log|p_n(z)| - \log\left|1 - \left(\frac{z}{r_n}\right)^{m_n}\right| + O(n).$$

Combined with the estimates (15) and (13) this shows that

$$\log|F(z)| \sim N(r)$$

outside small pits around the zeros.

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G. M. College, Sambalpur, India.

APPROXIMATE FUNCTIONAL APPROXIMATIONS AND THE RIEMANN HYPOTHESIS

ROBERT SPIRA

1. **Introduction.** Using the functional equation for the Riemann zeta function

$$\zeta(s) = \chi(s)\zeta(1-s) \tag{1}$$

where

$$1/\chi(s) = (2\pi)^{-s}\, 2\cos(\pi s/2)\Gamma(s), \tag{2}$$

it was shown in Spira [1] that

$$\zeta(s) \neq 0,\ \ 1/2 < \sigma < 1,\ \ t \geqq 10 \ \text{ implies } \ |\zeta(1-s)| > |\zeta(s)| \tag{3}$$

where $s=\sigma+it$. Using similar but improved techniques, Schoenfeld and Dixon [2] strengthened the result (3) to assuming only $\sigma>1/2$, $|t|\geqq 6.8$ and $\zeta(s)\neq 0$. It easily follows from this inequality that the Riemann hypothesis is equivalent to the inequality $|\zeta(1-s)| > |\zeta(s)|$, $1/2<\sigma<1$, $t\geqq 10$.

Consider now the formula for $\zeta(s)$ which gives rise to the approximate functional equation and the Riemann-Siegel formula:

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$$\zeta(s) = g_m(s) + \frac{e^{i\pi s}\Gamma(1-s)}{2\pi i}\int_C \frac{w^{s-1}e^{-mw}}{e^w - 1}\,dw \tag{4}$$

where

$$g_m(s) = \sum_{n=1}^{m} n^{-s} + \chi(s)\cdot \sum_{n=1}^{m} n^{s-1}. \tag{5}$$

The $g_m(s)$ are the approximate functional approximations of the title, and, as noted implicitly by Titchmarsh ([**4**, p. 74]), they satisfy the same functional equation as $\zeta(s)$. Hence, just as in the case of the ζ-function, $g_m(s)$ has its zeros on the critical line for $|t| > 6.8$ if and only if $|g_m(1-s)| > |g_m(s)|$.

It is thus natural to write (4) in the form

$$\zeta(s) = g_m(s) + B$$

and study the location of the zeros of $g_m(s)$, (hopefully on the critical line), and attempt to carry the final conclusion of the Riemann hypothesis via the ideas of Rouché.

It is indeed possible to show that $g_1(s)$ and $g_2(s)$ have their zeros on the critical line (for t sufficiently large) and this proof is carried out in §3, with the aid of two lemmas in §2.

Massive calculations were undertaken to verify the hypothesis for $m \geqq 3$, but these calculations instead revealed a remarkable scientific situation, which reinforces the possibility of using Rouché's theorem. The evidence strongly suggests the conjecture: If $m \geqq 3$, and s is in the critical strip, then $g_m(s)$ has its zeros on the critical line for $(2\pi m)^{1/2} \leqq t \leqq 2\pi m$, and has zeros off the line outside this interval. The computations supporting this conjecture will be reported in full in another paper.

2. **Lemmas on $\chi(s)$.** We write D for d/ds and D_σ for $\partial/\partial\sigma$.

Lemma 1. *If $|t| \geqq 10$ and $\sigma > 1/2$ then $D_\sigma \log|1/\chi(s)| > \log|s| - 1.93$.*

Proof. From Schoenfeld-Dixon [2], we have $D_\sigma \log |1/\chi(s)| > \log |s| - |s|^{-1}/2 - |s|^{-2}/12 - |t|^{-3}/5 - (\log 2\pi + \pi/(4 \sinh^2(\pi t/2)))$ from which the lemma easily follows.

Lemma 2. If *$|t| \geqq 10$ and $\sigma > 1/2$ then*

$$|1/\chi(s)| > .9646(|s|/(2\pi))^{\sigma-1/2}.$$

Proof. We have

$$|1/\chi(s)| = |2\pi|^{-\sigma}|2\cos(\pi s/2)||\Gamma(s)|. \tag{6}$$

As shown in Spira [1],

(7) $\quad |2\cos(\pi s/2)| \geqq 2\sinh(\pi t/2) = e^{\pi t/2} - e^{-\pi t/2} > .99e^{\pi t/2},$

the last inequality holding for $t \geqq 10$. Also from Spira [1] we have

(8) $\quad |\Gamma(s)| = (2\pi)^{1/2}e^{-\sigma}|s|^{\sigma-1/2}e^{-t\arg s}|e^{1/(12s)+R_1}|$

where $|R_1| < |s|^{-1}/6$. It is easy to see that if $|z| < 1$, then $|e^z| \geqq 1 - |z|[1/(1-|z|)]$. Now $|1/(12s)+R_1| < |s|^{-1}/12 + |s|^{-1}/6 = |s|^{-1}/4 \leqq 1/40$ if $t \geqq 10$. Hence, setting $z = 1/(12s)+R_1$,

(9) $\quad |e^{1/(12s)+R_1}| \geqq 1 - |z|[1/(1-|z|)] \geqq 38/39$

the last inequality holding since $|z| < 1/40$. By elementary geometry $t(\pi/2 - \arg s) > \sigma$, so the lemma follows on combining equations (6)–(9).

3. The cases $m = 1, 2$.

THEOREM *For $m = 1, 2$ and $|t|$ sufficiently large, $g_m(s)$ has all its complex zeros on $\sigma = 1/2$.*

PROOF. For $m = 1$ we have $g_m(s) = 1 + \chi(s)$, and for $\sigma > 1/2$ and $|t| > 6.8$, by Schoenfeld-Dixon [2], we have $|g_1(s)| \geqq 1 - |\chi(s)| > 0$. An easy argument shows that $g_1(s)$ has exactly one zero in each Gram interval.

For $m = 2$, $|g_m(s)| \geqq |1+2^{-s}| - |\chi(s)| \cdot |1+2^{s-1}|$, and $|g_2(s)| > 0$ provided

(10) $\quad |1/\chi(s)| > |(1+2^{s-1})/(1+2^{-s})|.$

On $\sigma = 1/2$ both sides of (10) are 1, so that proceeding as in Schoenfeld-Dixon [2], (10) will hold provided

(11) $\quad D_\sigma \log |1/\chi(s)| > D_\sigma \log |(1+2^{s-1})/(1+2^{-s})|.$

Since (Schoenfeld-Dixon [2]) $D_\sigma \log |f(s)| = \operatorname{Re} D \log f(s)$,

$$\begin{aligned} D_\sigma \log \left| \frac{1+2^{s-1}}{1+2^{-s}} \right| &= \log 2 \operatorname{Re}\left[\frac{1+2^{-s}+2^{s-1}}{(1+2^{-s})(1+2^{s-1})} \right] \\ &\leqq \log 2 \left| \frac{1+2^{-s}+2^{s-1}}{(1+2^{-s})(1+2^{s-1})} \right| \\ &\leqq \log 2 \left[\frac{1+2^{-\sigma}+2^{\sigma-1}}{(1-2^{-\sigma})(1-2^{\sigma-1})} \right] \end{aligned}$$

where we must now take $1/2 < \sigma < 3/4$ to obtain a bound on the denominator. The numerator $1+2^{-\sigma}+2^{\sigma-1}$ has a minimum at $\sigma = 1/2$, and for $1/2<\sigma<1$ rises monotonely from $1+(2)^{1/2}$ to 2.5. The denominator is $1.5-(2^{-\sigma}+2^{\sigma-1})$ which is smallest at $\sigma=3/4$. Thus $D_\sigma|\log(1+2^{s-1})/(1+2^{-s})|<2.5\log 2/[1.5-(2^{-3/4}+2^{-1/4})]<27$. Using Lemma 1, we need only choose $|s|$ so large that $\log|s|>1.93+27$, i.e., $t>e^{29}$.

For $\sigma>3/4$, we proceed directly from (10) using Lemma 2. We have $|(1+2^{s-1})/(1+2^{-s})|\leqq(1+2^{\sigma-1})/(1-2^{-3/4})$ so that (10) will hold provided

$$.9646(|s|/(2\pi))^{\sigma-1/2}>(1+2^{\sigma-1})/(1-2^{-3/4}). \tag{12}$$

For $3/4\leqq\sigma\leqq 1$, the right hand side of (12) is bounded by 5, and an easy calculation shows we need only take $t>2\pi\cdot 6^4\sim 8145$. For $\sigma>1$, $1+2^{\sigma-1}<2^\sigma$, so (12) transforms to $(|s|/4\pi)^{\sigma-1/2}>(2)^{1/2}/.376$, which will be valid if $t>4\pi((2)^{1/2}/.376)^4\sim 805$. This completes the proof of the theorem of this section.

Since there is empirically a steady appearance of zeros off the critical line for $m\geqq 3$, it appears unlikely that one would be able to extend the theorem of this section to any further m.

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University of Tennessee