A NOTE ON SPLITTING IN SOLVABLE GROUPS[1]

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1. **Introduction.** The theorem presented below generalizes theorems of E. Schenkman [5] and G. Higman [4] concerning splitting in finite solvable groups. This generalization is achieved by applying results from the theory of formations recently developed by W. Gaschütz [2], [3]. All groups considered here are finite and solvable. A *formation*, $\mathfrak{F}$, is a collection of groups closed under taking homomorphisms and subdirect products. It follows that every group, G, contains a characteristic subgroup, $G_{\mathfrak{F}}$, minimal with respect to the property that $G/G_{\mathfrak{F}} \in \mathfrak{F}$. A formation, $\mathfrak{F}$, is called *saturated*, if $G/\phi(G) \in \mathfrak{F}$ implies $G \in \mathfrak{F}$ for all G (see [3]). F is called an $\mathfrak{F}$-subgroup of G if $F \in \mathfrak{F}$ and if $F \subseteq H \subseteq G$ implies $FH_{\mathfrak{F}} = H$. A theorem of Gaschütz [2], states that if $\mathfrak{F}$ is saturated, $\mathfrak{F}$-subgroups of G always exist and are conjugate in G.

THEOREM. *Let $\mathfrak{F}$ be a saturated formation and suppose that for a finite solvable group, G, $G_{\mathfrak{F}}$ is abelian. Then:*

(i) *the $\mathfrak{F}$-subgroups of G complement $G_{\mathfrak{F}}$.*

(ii) *all complements of $G_{\mathfrak{F}}$ in G are conjugate and hence are $\mathfrak{F}$-subgroups of G.*

Let $L_0(G) = G$ and let $L_i(G)$ be the ith term of the lower nilpotent series of G. If $\mathfrak{F}$ denotes the formation of groups having nilpotent length $\leqq k-1$, the theorem yields a theorem of Higman [4] which states that if $L_k(G)$ is abelian, G splits over $L_k(G)$ and all complements of $L_k(G)$ in G are conjugate. (This statement becomes a theorem of Schenkman [5] when $k=2$.) R. Carter [1] was able to identify the complements in Higman's theorem as the relative system normalizers of $L_{k-1}(G)$ in G. From (ii) we may also identify them as the $\mathfrak{F}$-subgroups of G (or as the Carter subgroups of G when $k=2$). Our theorem yields a number of other interesting results on splitting when $\mathfrak{F}$ ranges over various saturated formations, for example, groups having nilpotent commutator subgroups, supersolvable groups, groups G, for which G/G' is a π-group, etc.

2. **Preliminary results and proof of the theorem.** Our proof employs a number of basic results of Gaschütz. First if F is an $\mathfrak{F}$-subgroup

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of G, $F \subseteq H \subseteq G$ implies that F is an $\mathfrak{F}$-subgroup of H. Also if N is normal in G, FN/N is an $\mathfrak{F}$-subgroup of G/N and every $\mathfrak{F}$-subgroup of G/N has the form F_0N/N where F_0 is an $\mathfrak{F}$-subgroup of G. Let π and π' denote a partition of the set of primes. $O_{\pi'}(G)$ denotes the maximal normal π'-subgroup of G and $O_{\pi'\pi}(G)$ denotes the subgroup of G for which $O_{\pi'\pi}(G)/O_{\pi'}(G)$ is the maximal normal π-subgroup of $G/O_{\pi'}(G)$. A formation, $\mathfrak{F}$, is said to be *locally defined* if for some sequence of (possibly empty) formations, $\mathfrak{f}(p)$, p ranging over the primes, $G \in \mathfrak{F}$ if and only if $p \nmid |G|$ when $\mathfrak{f}(p)$ is empty and $G/O_{p'p}(G) \in \mathfrak{f}(p)$ otherwise. Gaschütz proved [2] that all locally defined formations are saturated and recently has announced[2] the important result that, conversely, all saturated formations are locally defined by *some* sequence of local formations $\{\mathfrak{f}(p)\}$.

Proof of the theorem. (i) If $G_{\mathfrak{F}} = E$, G is its own $\mathfrak{F}$-subgroup and the theorem is trivial. Suppose then that $G_{\mathfrak{F}} \neq E$. Let F be an $\mathfrak{F}$-subgroup of G. Since $FG_{\mathfrak{F}} = G$, to prove (i) it suffices to show that $F \cap G_{\mathfrak{F}} = E$. Suppose $F \cap G_{\mathfrak{F}} \neq E$. Then since $G_{\mathfrak{F}}$ is abelian, $F \cap G_{\mathfrak{F}}$ is normal in G. Let N be a minimal normal subgroup of G. Then $(G/N)_{\mathfrak{F}} = G_{\mathfrak{F}}N/N$ is abelian and FN/N is an $\mathfrak{F}$-subgroup of G/N. By induction, $FN \cap G_{\mathfrak{F}}N \subseteq N$. Thus $F \cap G_{\mathfrak{F}}$ lies in every minimal normal subgroup of G. It follows that $F \cap G_{\mathfrak{F}} = N_0$, is the unique minimal normal subgroup of G.

Suppose $N_0 = G_{\mathfrak{F}}$. Then $G = FN_0 = F \in \mathfrak{F}$ whence $G_{\mathfrak{F}} = E$, a contradiction. Hence $N_0 \subset G_{\mathfrak{F}}$.

Suppose $F \subset H \subset G$. Then $HG_{\mathfrak{F}} = G$ and so $G/G_{\mathfrak{F}} \simeq H/(H \cap G_{\mathfrak{F}}) \in \mathfrak{F}$ whence $H_{\mathfrak{F}} \subset G_{\mathfrak{F}}$. Since $H_{\mathfrak{F}}$ is now forced to be abelian and F is an $\mathfrak{F}$-subgroup of H, $F \cap H_{\mathfrak{F}} = E$ by induction on H. On the other hand the fact that $G_{\mathfrak{F}}$ is abelian implies $H_{\mathfrak{F}}$ is normal in G and hence $H_{\mathfrak{F}} \cap F$ contains N_0, a contradiction. Thus F is maximal in G and $G_{\mathfrak{F}}/N_0$ and N_0 are successive chief factors of G. From the uniqueness of N_0, $G_{\mathfrak{F}}$ is an abelian p-group.

Let Q be a p'-subgroup of G such that $QG_{\mathfrak{F}}$ is normal in G. Then, since $G_{\mathfrak{F}}$ is abelian, $G_{\mathfrak{F}} = C_{G_{\mathfrak{F}}}(Q) \times [Q, G_{\mathfrak{F}}]$ where each component is normal in G. Suppose $[Q, G_{\mathfrak{F}}] \neq G_{\mathfrak{F}}$. Then uniqueness of N_0 implies $C_{G_{\mathfrak{F}}}(Q) = G_{\mathfrak{F}}$ and Q is then normal in G. Because of the uniqueness of N_0, $Q = E$. Thus if $QG_{\mathfrak{F}} \Delta G$ either $Q = E$ or $G_{\mathfrak{F}} = [Q, G_{\mathfrak{F}}]$.

Choose B_q so that $B_q/N_0 = O_{q'q}(G/N_0)$, and set $T_q = O_{q'q}(F)$. We shall show that $T_q \subseteq B_q$.

Suppose $q \neq p$. Then $G_{\mathfrak{F}} \subseteq O_{q'}(G)$, and $T_qG_{\mathfrak{F}}$ is q-nilpotent and normal in G. Hence $T_qG_{\mathfrak{F}} \subseteq O_{q'q}(G) \subseteq B_q$.

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Suppose $q=p$. Set $Q_0=O_{p'}(F)$. Since $Q_0G_{\mathfrak{F}}$ is normalized by both F and $G_{\mathfrak{F}}$, it is normal in G. By a previous remark, if $Q_0\neq E$, $G_{\mathfrak{F}}=[Q_0, G_{\mathfrak{F}}]$. But the latter is impossible since $Q_0\subseteq O_{p'}(F)$ and $N_0\subseteq O_p(F)$ imply $[Q, N_0]=E$. Thus $Q_0=E$. As a result, $T_p=O_p(F)$ and $T_pG_{\mathfrak{F}}$, being normalized by F and $G_{\mathfrak{F}}$, lies in $O_p(G)\subseteq B_p$. Hence $T_p\subseteq B_p$.

Since $\mathfrak{F}$ is saturated, we may assume $\mathfrak{F}$ is locally defined by $\{\mathfrak{f}(p)\}$. Thus $F\in\mathfrak{F}$ implies $F/T_q\in\mathfrak{f}(q)$ for each prime q dividing $|F|$. Since $T_q\subseteq B_q$, it follows that $G/B_q\simeq F/(F\cap B_q)$ is a homomorphic image of F/T_q. Thus $G/B_q\in\mathfrak{f}(q)$ for each prime, q, dividing $[G: N_0]$. Since $\mathfrak{F}$ is locally defined by $\{\mathfrak{f}(q)\}$, $G/N_0\in\mathfrak{F}$ whence $G_{\mathfrak{F}}\subseteq N_0$, a contradiction, and (i) is proved.

(ii) In proving the second part of the theorem it suffices to show that every complement, K, of $G_{\mathfrak{F}}$ in G, is an $\mathfrak{F}$-subgroup of G. Again, there is nothing to prove if $G_{\mathfrak{F}}=E$. We suppose that $G_{\mathfrak{F}}\neq E$. Let K be an arbitrary complement of $G_{\mathfrak{F}}$ in G and choose N minimal normal in G contained in $G_{\mathfrak{F}}$. Then $KN\cap G_{\mathfrak{F}}=(K\cap G_{\mathfrak{F}})N=N$ so KN/N is a complement of $G_{\mathfrak{F}}/N=(G/N)_{\mathfrak{F}}$ in G/N. By induction KN/N is an $\mathfrak{F}$-subgroup of G/N and so $KN=FN$ where F is an $\mathfrak{F}$-subgroup of G.

Suppose $N\subset G_{\mathfrak{F}}$. Then $KN=FN\subset G$. Now from (i), $F\cap N\subseteq F\cap G_{\mathfrak{F}}=E$ and so $F\subset FN$. Consequently, $(FN)_{\mathfrak{F}}$, being a nontrivial characteristic subgroup of N must coincide with N. Since K complements $(KN)_{\mathfrak{F}}=N$ in KN, induction on KN yields that K is an $\mathfrak{F}$-subgroup of KN. Since F is an $\mathfrak{F}$-subgroup of KN as well as G, K and F are conjugate in KN. Thus K is an $\mathfrak{F}$-subgroup of G.

Suppose $N=G_{\mathfrak{F}}$. Then $K\in\mathfrak{F}$ and K is maximal in G. Under these circumstances K satisfies the defining properties of an $\mathfrak{F}$-subgroup of G.

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