A PARTIAL GEOMETRY $(q^3+1, q^2+1, 1)$ AND CORRESPONDING PBIB DESIGN

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In [1] Bose introduced the concept of a *partial geometry* (r, k, t) which is a system of undefined points and lines together with an incidence relation satisfying:

A1. Any two points are incident with not more than one line.

A2. Each point is incident with r lines.

A3. Each line is incident with k points.

A4. If the point P is not incident with the line L, there pass through P exactly t lines intersecting L.

A *PBIB design* $(r, k, \lambda_1, \lambda_2)$ is an arrangement of r objects into b sets (called blocks) such that:

B1. Each object is contained in exactly r sets.

B2. Each block contains k distinct objects.

B3. A pair of objects occur together either λ_1 times or λ_2 times. Those occurring together λ_1 times are called *first associates*, those occurring together λ_2 times are called *second associates*.

From a partial geometry (r, k, t) a PBIB design $(r, k, 1, 0)$ may be constructed where the objects and blocks are the points and lines, respectively, of the partial geometry and where two objects are first associates if the corresponding points are collinear, second associates otherwise.

In [1] two nontrivial examples of partial geometries are given. One is the points and lines of a nondegenerate quadric surface in projective 4-space, $P(4, q)$ over the field $GF(q)$ of q elements. The parameters (r, k, t) are $(q+1, q+1, 1)$. The explicit construction and corresponding PBIB design $(q+1, q+1, 1, 0)$ are given in [2]. The second example is the points and lines of a nondegenerate elliptic quadric surface in $P(5, q)$. The corresponding PBIB design is given in [3].

The purpose of this paper is to prove the following theorem.

THEOREM 1. *Consider the set H of points (x_i), $-2 \leqq i \leqq 2$, in $P(5, q^2)$ satisfying $x_{-2}\bar{x}_2+x_{-1}\bar{x}_1+x_0\bar{x}_0+x_1\bar{x}_{-1}+x_2\bar{x}_{-2}=0$ where $\bar{x}_i$ is the image of x_i under the automorphism of order two of $GF(q^2)$ leaving $GF(q)$ fixed. Then the points and lines of H form a partial geometry $(q^3+1, q^2+1, 1)$.*

For y in H, the points x in H lying on lines of H through y are precisely those points in the tangent hyperplane to H through y. Using the fact that $\bar{y}_i = y_i^q$ this hyperplane is found upon differentiation to be given by

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$$\bar{y}_{-2}x_2 + \bar{y}_{-1}x_1 + \bar{y}_0x_0 + \bar{y}_1x_{-1} + \bar{y}_2x_{-2} = 0. \tag{1}$$

Now there are clearly q^2+1 points on each line of H since the lines are in $P(5, q^2)$. Consider the point $y=(1, 0, 0, 0, 0)$ in H. In this case (1) becomes

$$x_2 = 0 \tag{2}$$

and thus the points of H collinear with y satisfy (2) and

$$x_{-1}\bar{x}_1 + x_0\bar{x}_0 + x_1\bar{x}_{-1} = 0. \tag{3}$$

Now it is well known (see, for example, [4]) that there are $1+q^2+q^5$ points of $P(5, q)$ satisfying (2) and (3). Since there are $1+q^2$ points on each line, this means there are $1+q^3$ lines through y. But it is shown in [4] that the 5 dimensional unitary group is transitive on the points of H. Thus there are $1+q^3$ lines of H through every point of H. Now consider the line L in H given by points $(\lambda, \mu, 0, 0, 0)$, λ, μ in $GF(q^2)$. We show now that L is not a side of any triangle whose points and lines are in H. Indeed if it were, there would exist a point x of H not on L collinear with two points on L; i.e. there would exist λ, λ', μ, μ' with $\lambda\mu'-\lambda'\mu\neq 0$ such that P would be collinear with $(\lambda, \mu, 0, 0, 0)$ and $(\lambda', \mu', 0, 0, 0)$. Thus from (1) the components of x satisfy

$$\begin{aligned} \bar{\lambda}x_2 + \bar{\mu}x_1 &= 0, \\ \bar{\lambda}'x_2 + \bar{\mu}'x_1 &= 0, \end{aligned} \tag{4}$$

which imply that $x_1=x_2=0$ since the coefficient determinant is nonzero. Since x is in H we have also $x_0=0$. But then x is on L, a contradiction. It is shown in [4] that the 5 dimensional unitary group is transitive on the lines of H. Therefore no line is contained in a triangle of H. We have now shown that for a nonincident point line pair P, L' there is at most one line through P intersecting L'. To complete the proof of Theorem 1 it is sufficient to show there is at least one line through P intersecting L'. Now there are $1+q^2$ points on L', and through each point of L' there are q^3 further lines, each of which contains q^2 further points. Thus there are $(1+q^2)q^3$ points not on L' but lying on a line intersecting L'. All these points are different since otherwise a triangle in H would be formed. However it is well known that there are $(1+q^2)$ $(1+q^5)$ total points in H and so all points in H not on L' lie on a line intersecting L'. The proof is complete.

Since the role of points and lines in a partial geometry may be interchanged, a $(q^2+1, q^3+1, 1)$ partial geometry has also been constructed.

References

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