COMPLETELY 0-SIMPLE AND HOMOGENEOUS n REGULAR SEMIGROUPS

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1. The purpose of this paper is to generalize R. McFadden and Hans Schneider's Theorem [**3**].

2. **Definition and notation.** Let $a \neq 0$ be a regular element of a semigroup S. An element x in S is called an inverse of a if $axa=a$ and $xax=x$. Let n be a fixed positive integer. A semigroup S with zero is said to be homogeneous n regular if every nonzero element of S has precisely n distinct inverse elements in S. A semigroup S with zero is said to be null if $SS=\{0\}$. A semigroup S will be called a right (left) zero semigroup if $xy=y$ $(xy=x)$ for all x, y in S. $|T|$ denotes the cardinality of a set T.

If S is completely 0-simple, I shall follow Clifford-Preston [**1**] (with J replacing Λ) and let $\{R_i: i \in I\}$ be the set of nonzero R-classes, $\{L_j: j \in J\}$ the set of nonzero L-classes, $\{H_{ij}=R_i \cap L_j: (i, j) \in (IxJ)\}$, be the set of nonzero H-classes and write $R_i^0=R_i \cup \{0\}$. If $a \neq 0$ is in a semigroup S, $E_a=\{e \in S: e=e^2 \text{ and } ea=a\}$, $F_a=\{f \in S: f=f^2 \text{ and } af=a\}$, $N_a=\{x \in S: axa=a \text{ and } xax=x\}$, $h(i) = |\{j \in J: H_{ij} \text{ is a group}\}|$ and $k(j)=|\{i \in I: H_{ij} \text{ is a group}\}|$. If $T \subseteq S$, $\mathcal{E}(T)=\{e \in T: e=e^2\}$. A homogeneous n regular semigroup S is called an (h, k) type if for all $a \in S \backslash 0$, $|E_a|=h$ and $|F_a|=k$, where h and k are fixed positive integers with $hk=n$.

3. We shall need the following lemmas.

LEMMA A. *Let S be completely 0-simple. If $a \in H_{ij}$, then*
(1) $E_a=\mathcal{E}(R_i)$ *and* $|E_a|=h(i)$.
(2) $F_a=\mathcal{E}(L_j)$ *and* $|F_a|=k(j)$.
(3) $|N_a|=h(i)k(j)$.

PROOF. (1) Since H_{ik}, $k \in J$ contains an idempotent if and only if H_{ik} is a group, $|\mathcal{E}(R_i)|=h(i)$. By Lemma 2.14 of [**1**], $\mathcal{E}(R_i) \subseteq E_a$. If $e \in E_a$, then obviously $\{0\} \neq eS \subseteq aS \subseteq R_i^0$ whence $E_a \subseteq \mathcal{E}(R_i)$. Hence $E_a=\mathcal{E}(R_i)$ and (1) follows. The proof of (2) is similar.

As an immediate application of [**1**, Theorem 2.18], we see that $a \in H_{ij}$ has an inverse in H_{mn} if and only if both H_{mj} and H_{in} are groups, and in this case the inverse in H_{mn} is unique. Thus (3) follows.

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LEMMA B. *For all a, b in a completely* 0-*simple semigroup S $aba = a \neq 0$ implies $bab = b$.*

PROOF. Let $axa = a \neq 0$. Then $\{0\} \neq S(ax) = Sx$ whence ax is a right identity of Sx and $x \in Sx$. Hence $xax = x$.

4. The following theorem is a generalization of R. McFadden and Hans Schneider's Theorem.

THEOREM. *Let S be a* 0-*simple semigroup and let n be a positive integer. Then there exist positive integers h and k such that $n = hk$ and for which the following statements are equivalent.*

(i) *S is an (h, k) type homogeneous n regular and completely* 0-*simple semigroup.*

(ii) *For every $a \neq 0$ in S there exist precisely n distinct nonzero elements $\{x_i\}_{i=1}^n$, such that $ax_ia = a$ for $i = 1, 2, \cdots, n$, and for c, d in S $cdc = c \neq 0$ implies $dcd = d$.*

(iii) *For every $a \neq 0$ in S there exist precisely h distinct nonzero idempotents $\{e_i\}_{i=1}^h = E_a$ and k distinct nonzero idempotents $\{f_j\}_{i=1}^k = F_a$ such that $E_a \cap F_a$ contains at most one element.*

(iv) *Every nonzero principal right ideal R contains precisely h nonzero idempotents and every nonzero principal left ideal L contains precisely k nonzero idempotents such that $R \cap L$ contains at most one nonzero idempotent.*

(v) *S is completely* 0-*simple. For every* 0-*minimal right ideal R there exist precisely h* 0-*minimal left ideals $\{L_i\}_{i=1}^h$ and for every* 0-*minimal left ideal L there exist precisely k* 0-*minimal right ideals $\{R_j\}_{j=1}^k$ such that $LR_j = L_iR = S$, for every $i = 1, 2, \cdots, h$, $j = 1, 2, \cdots, k$.*

(vi) *S is completely* 0-*simple. Every* 0-*minimal right ideal R is the union of a right group with zero G^0, a union of h disjoint groups except zero, and a null subsemigroup Z which annihilates the right ideal R on the left and every* 0-*minimal left ideal L is the union of a left group with zero G'^0, a union of k disjoint groups except zero, and a null subsemigroup Z' which annihilates the left ideal L on the right.*

(vii) *S contains at least n nonzero distinct idempotents, and for every nonzero idempotent e there exists a set $E = \{e_i\}_{i=1}^n$ of nonzero idempotents of S such that eE is a right zero subsemigroup of S containing precisely h nonzero idempotents, Ee is a left zero subsemigroup of S containing precisely k nonzero idempotents, $e(\mathcal{E}(S) \backslash E) = \{0\} = (\mathcal{E}(S) \backslash E)e$, and $eE \cap Ee = \{e\}$.*

REMARK 1. If $n = 1$, then the theorem above takes the same form as R. McFadden and Hans Schneider's Theorem [3].

5. **Proof of the theorem.** (i) implies (ii). This is clear by the definition of an (h, k) type homogeneous n regular semigroup and Lemma B. (ii) implies (iii). We shall prove the existence of a nonzero primitive idempotent of S. Let a be a nonzero element of S. By (ii) there exist $\{x_i\}_{i=1}^n$ in S such that $ax_ia=a$ and $x_iax_i=x_i$ for every $i=1, 2, \cdots, n$.

Choose $ax_1=e$. Clearly $0\neq e\in \mathcal{E}(S)$. Let f be any nonzero idempotent such that $fe=ef=f$. Since $fef=(fe)f=ff=f$, we have $efe=e$ by the assumption of (ii). But we have $efe=e(fe)=ef=f$. Hence we conclude $e=f$, and e is a nonzero primitive idempotent of S and hence S is completely 0-simple [**1**, p. 76]. The last assertion of (iii) now follows since each H-class has at most one idempotent.

Let $a\in H_{ij}$ and $b\in H_{mq}$. Define $h=h(m)$ and $k=k(q)$. Let $c\in H_{iq}$ and $d\in H_{mi}$. By Lemma A and (ii)

$$\begin{aligned} n &= |N_a| = h(i)k(j), \\ n &= |N_c| = h(i)k(q) = h(i)k, \\ n &= |N_b| = h(m)k(q) = hk. \end{aligned}$$

Thus it follows that $h=h(i)$, $k=k(j)$, $|E_a|=h$, and $|F_a|=k$. (iii) implies (iv). By (iii), S contains nonzero idempotent. Let e, f be nonzero idempotents such that $ef=fe=f$. Then both e, f are in $E_f\cap F_f$, whence $e=f$. Hence S is completely 0-simple. The rest is just Lemma A, parts (1), (2).

(iv) implies (v). By (iv), it is clear that S has a nonzero primitive idempotent, and hence S is completely 0-simple. Then every nonzero principal right ideal $R(a)=a\cup aS=aS$ for $a\neq 0$ in S is a 0-minimal right ideal of S by Exercise 2 in [**1**, p. 83]. Let $\mathcal{E}(R(a)\backslash 0)=\{e_i\}_{i=1}^h$ and let $L_i=Se_i$. Then $\{L_i\}_{i=1}^h$ are 0-minimal left ideals of S such that $L_iR(a)=S$ $(i=1, 2, \cdots, h)$ by [**1**, Lemma 2.46]. The proof for a 0-minimal left ideal $L(a)=Sa$ is analogous. (v) implies (vi). Let R be a 0-minimal right ideal of S. Then by (v) there exists a set $\{L_i\}_{i=1}^h$ of 0-minimal left ideals such that $L_iR=S$ $(i=1, 2, \cdots, n)$. By [**1**, Lemma 2.46], $R\cap L_i=RL_i$ is a group with zero. Let $G^0=\bigcup_{i=1}^h (RL_i)$ and let Z be the complement of the nonzero part of G^0 in R. Then $R=G^0\cup Z$, and $ZR=\{0\}$ since each element of Z belongs to a 0-minimal left ideal L' for which $L'R=\{0\}$ by [**1**, Lemma 2.46]. Therefore Z is a null subsemigroup of S. By [**1**, Exercise 2, p. 39], it suffices to show that $\mathcal{E}(G)=\mathcal{E}(G^0\backslash 0)$ is a right zero semigroup. From [**1**, Lemma 2.43], it follows that $\mathcal{E}(R\backslash 0)$ is a right zero semigroup, and so is $\mathcal{E}(G)$. The proof of the rest is similar to the proceeding argument.

(vi) implies (vii). Assume (vi). Let $e\in \mathcal{E}(S\backslash 0)$ and let $\mathcal{E}(eS\backslash 0)$

$=\{e_i\}_{i=1}^{h}$. Define $E=\bigcup_{i=1}^{h}(Se_i\backslash 0)$. Then $|E|=hk=n$. From $Ee\subset Se$ and $Ee\subseteq \mathcal{E}(Se)$ it follows that Ee is a left zero semigroup with $|Ee|=k=|\mathcal{E}(Se\backslash 0)|$. We claim that $(\mathcal{E}(S)\backslash E)\cdot e=\{0\}$.

Assume, by way of contradiction, that $ge\neq 0$ for some g in $(\mathcal{E}(S)\backslash E)$. Setting $L=Sg$ and $R=eS$, we have that $RL=R\cap L$ is a group with zero by Lemma 2.46, [1]. Then $g\in\mathcal{E}(L)\subseteq E$, contrary to $g\in\mathcal{E}(S)\backslash E$. Thus we must have $(\mathcal{E}(S)\backslash E)\cdot e=\{0\}$. Analogously, we can show that $e\cdot E$ is a right zero semigroup, $|eE|=h$ and $e\cdot(\mathcal{E}(S)\backslash E)=\{0\}$. Finally, from $(eE\cap Ee)\subset(eS\cap Se)=H_e^0$, it follows that $eE\cap Ee=\{e\}$. (vii) implies (i). If $ef=fe=f$ $0\neq f=f^2$ then $f\in E$ by $e(\mathcal{E}(S)\backslash E)=\{0\}$, whence $f\in eE\cap Ee=\{e\}$. Thus $e=f$, and S is completely 0-simple. Suppose $eS\backslash 0=R_i$. Since $e(\mathcal{E}(S)\backslash E)=\{0\}$, it follows that $\mathcal{E}(R_i)\subseteq E$, whence $\mathcal{E}(R_i)\subseteq eE$. But as eE is a right zero subsemigroup each $g\in eE$ is idempotent. Also $0\notin eE$, for since $e\in eE$, $xe=e$, all $x\in eE$. Hence $eE\subseteq\mathcal{E}(R_i)$. We have proved that $\mathcal{E}(R_i)=eE$. Let $0\neq a\in H_{ij}$. There exists an idempotent $e\in R_i$. Then $eE=\mathcal{E}(R_i)=E_a$, by Lemma A, whence $|E_a|=|eE|=h$ by (vii). Similarly, $|F_a|=k$. By Lemma A, $|N_a|=h\cdot k=n$ and (i) is proved.

This completes the proof of the theorem.

Remark 2. In the theorem above, h, k, and n could be infinite cardinals.

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