GENERALIZED DERIVATIVES AND THE DE LA VALLÉE POUSSIN DERIVATIVE¹

C. KASSIMATIS

Let $V_n[f; p_1, \dots, p_{n+1}]$ denote the n th divided difference of f relative to the distinct points $p_1, \dots, p_{n+1}$, where f is a real valued continuous function over the interval (a, b) and $p_i \in (a, b)$, $i=1, \dots, n+1$.

If

$$\lim_{h \rightarrow 0} n! V_n[f; x + x_1 h, \dots, x + x_{n+1} h]$$

exists and is finite, it is called, following Denjoy [1, p. 305], the n th E -generalized derivative of f at x , $f_{n,E}(x)$, where $E = \{x_1, \dots, x_{n+1}\}$. If $f_{n,E}(x)$ is independent of the choice of $E \subset S$ for a subset S of the reals, we will call it the n th S -generalized derivative $f_{n,S}(x)$.

If

$$(1) \quad f(x+t) = a_0 + a_1 t + \dots + a_n t^n / n! + o(t^n)$$

where the numbers $a_0, a_1, \dots, a_n$ depend on x only, then a_n will be called, following Marcinkiewicz and Zygmund [5, p. 1], the n th de la Vallée Poussin derivative of f at x , $f_{(n)}(x)$.

Denjoy has shown [1, p. 289] that the existence of $f_{(n)}(x)$ implies that of $f_{n,S}(x)$ for S arbitrary. In a previous paper [4] the author has obtained a converse of the above Denjoy theorem. The purpose of the present paper is to establish a more general form of this converse theorem.

Let f and S be as above. Denote by S_{j+1} any $(j+1)$ -tuple of distinct points of S , $j=1, \dots, n-1$. For $x \in (a, b)$ and $s_i \in S$, set $y_i = x + s_i h$, $i=1, \dots, n+1$.

THEOREM. Suppose that $f_{n,S}(x)$ exists. If

- (i) $f_{j,S_{j+1}}(x)$ exists for all $j=1, \dots, n-1$,
- (ii) $\lim_{h \rightarrow 0} \{ (y - y_{n+1}) V_{n+1}[f; y_1, \dots, y_{n+1}, y] \} = \theta(x, y - x)$ where $\theta(x, y - x) \rightarrow 0$ as $y \rightarrow x$, then $f_{(n)}(x)$ exists and $f_{(n)}(x) = f_{n,S}(x)$.

PROOF. By [3, Corollary 3] we obtain the existence of $f_{1,S}(x)$, $\dots, f_{n-1,S}(x)$. Consequently, $f_{1,E_2}(x), \dots, f_{n,E_{n+1}}(x)$ exist, where $E_i = \{s_1, \dots, s_i\}$ for $i=1, \dots, n+1$, and the result follows by [4, Theorem].

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REMARKS. (i) As mentioned earlier, Denjoy has shown that the existence of $f_{(n)}(x)$ implies that of $f_{n,s}(x)$ for S arbitrary. Since the existence of $f_{(n)}(x)$ implies by (1) that of $f_{(1)}(x), \dots, f_{(n-1)}(x)$, it follows that the existence of $f_{(n)}(x)$ implies the first condition of the above theorem.

(ii) The existence of $f_{(n)}(x)$ implies also the second condition of the above theorem. Indeed in this case, the expression $f(x+t) + o(t^n)$ is by (1) a polynomial of degree at most n and it follows by [2, Theorem 2.1] that

$$0 = V_{n+1}[f(x+t) + o(t^n): s_1h, \dots, s_{n+1}h, y-x].$$

Hence,

$$\begin{aligned} (2) \quad V_{n+1}[f(t): x + s_1h, \dots, x + s_{n+1}h, y] \\ = V_{n+1}[o(t^n): s_1h, \dots, s_{n+1}h, y-x]. \end{aligned}$$

Further, evaluating the divided difference we get:

$$\begin{aligned} (y - y_{n+1})V_{n+1}[o(t^n): s_1h, \dots, s_{n+1}h, y-x] \\ = \frac{(y - y_{n+1})o((y-x)^n)}{(y - y_1) \cdots (y - y_{n+1})} \\ - \sum_{i=1}^{n+1} \frac{(y - y_{n+1})o(s_i^n h^n)}{h^n(y - y_i)\{(u - s_1) \cdots (u - s_{n+1})\}'_{u=s_i}}. \end{aligned}$$

When $h \rightarrow 0$, we obtain

$$\lim_{h \rightarrow 0} \{(y - y_{n+1})V_{n+1}[o(t^n): s_1h, \dots, s_{n+1}h, y-x]\} = \theta(x, y-x).$$

The result follows from relation (2).

(iii) The following example shows that the first condition of the above theorem cannot alone ensure the existence of $f_{(n)}(x)$. Consider the continuous function $f(x) = x|x|$. Let $n=2$, $x=0$, $s_1=-1$, $s_2=0$, $s_3=1$. Evaluating the generalized derivatives we find:

$$f_{1,[-1,0]}(0) = 0; \quad f_{2,\{-1,0,1\}}(0) = 0.$$

For $y > 0$, we find that

$$\lim_{h \rightarrow 0} \{(y-h)V_3[x|x| : -h, 0, h, y]\} = 1 \neq \theta(0, y-0).$$

The de la Vallée Poussin derivative $f_{(2)}(0)$ does not exist because otherwise we must have

$$t|t| = f_{(1)}(0)t + f_{(2)}(0)t^2/2 + o(t^2),$$

whence $f_{(1)}(0) = o(t^2) = 0$. Therefore

$$t | t | = f_{(2)}(0)t^2/2.$$

From this relation we obtain $f_{(2)}(0) = 2$ when $t > 0$, and $f_{(2)}(0) = -2$ when $t < 0$. Since $f_{(2)}(0)$ here depends on t , the second de la Vallée Poussin derivative of $x|x|$ does not exist at the origin.

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WAYNE STATE UNIVERSITY