INNER FACTORS AND BLASCHKE PRODUCTS[1]

NORMAN RUSHFIELD

1. Introduction. Let $L_{2,v}$ denote the space of functions on the positive real axis into a separable Hilbert space V, such that $\int_0^\infty |f(x)|^2 dx < \infty$ and for every $u \in V$, $(f(x), u)$ is measurable. Let $\hat{L_{2,v}}$ denote the space of Fourier transforms

$$\hat{f}(t) = \int_0^\infty f(x) e^{itx}\, dx,$$

where $f \in L_{2,v}$ and dx denotes $dx/2\pi$. The functions in $\hat{L_{2,v}}$ can be extended to analytic functions in the upper half plane. We denote this space of analytic functions by H_v. By the Paley-Wiener theorem, H_v is characterized by the property that $h \in H_v$ if and only if for some constant M and every $y > 0$, $\int_{-\infty}^{\infty} (h(x+iy), h(x+iy)) dx < M$. H_v is a Hilbert space with inner product $(f, g)_1 = \int_0^\infty (f(x), g(x)) dx$.

Let T_s (for fixed $s > 0$) be the left translation operator on $L_{2,v}$, $T_s f(x) = f(x+s)$. The family $\{T_s \mid s > 0\}$ is a semigroup of operators.

Let τ_s (for fixed $s > 0$) be the right translation operator on $L_{2,v}$:

$$\tau_s g(x) = \begin{cases} g(x-s), & \text{for } x - s \geqq 0, \\ 0, & \text{for } x - s < 0. \end{cases}$$

The family $\{\tau_s \mid s > 0\}$ is a semigroup of isometric operators.

A subspace l of $L_{2,v}$ is said to be *left invariant* (an l-space) if for every $f \in l$, $\{T_s f \mid s > 0\} \subseteq l$.

A subspace r of $L_{2,v}$ is said to be *right invariant* (an r-space) if for every $g \in r$, $\{\tau_s g \mid s > 0\} \subseteq r$. It is easily seen that the orthogonal complement of an r-space is an l-space and vice-versa.

An *inner factor* is an operator valued function defined and analytic in the upper half plane such that for each z, $F(z): W \to V$ (where W and V are separable Hilbert spaces), $\|F(z)\| \leqq 1$, and for almost all real z, $F(z)$ is an isometry. If V is finite dimensional, and $W = V$ then $\det F$ (defined as the determinant of the matrix $(F(z)u_i, u_i)$, where u_i is an orthonormal basis for V) is a scalar inner factor.

Let R denote the Fourier transform space of an r-space. An R-space is characterized as being invariant under multiplication by

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e^{isz} for every $s>0$. The following result has been proved by Lax [4], [5] and by Halmos [2]:

THEOREM A. *Every (nonzero) closed R-space can be represented in the form* $R_v = F^v H_w$, *where* F^v *is an inner factor and* $F^v(z): W \to V$.

The inner factor corresponding to an R-space is unique to within multiplication on the right by a constant unitary operator. By regarding two inner factors as equivalent if they differ on the right by a constant unitary operator, we obtain a one-to-one correspondence between nonzero closed R-spaces and inner factors. This correspondence carries over to closed l-spaces ($l \subset L_{2,v}$), where R is the Fourier transform space of r ($=l^{\perp}$) and will be denoted by a common subscript $l_a \sim F_a$ (or lack of subscript, $l \sim F$). From the division theory of [4], we have

THEOREM B. $l_a \subseteq l_b$ *if and only if there exists an inner factor* F_c *such that* $F_b = F_a F_c$.

A *(generalized) exponential* is a function $f \in L_{2,v}$ of the form $f = p(x)e^{i\lambda x}$ where $p(x)$ is a polynomial with coefficients in V, and $\operatorname{Im} \lambda > 0$. We define the order of the exponential f as the degree of $p(x)$; λ is called the exponent belonging to the exponential f.

DEFINITION. An inner factor F is said to be a Blaschke product if l ($\sim F$) is spanned by the exponentials contained in l.

We show that in the case of scalar inner factors, our definition of a Blaschke product is equivalent to the standard definition.

Let $[v_j | j \in J]$ denote the smallest closed l-space containing $\{v_j | j \in J\}$. Let $l = [x^{k_j} e^{i\lambda_j x} | j \in J]$; we note that

$$[x^{k_j} e^{i\lambda_j x} \,|\, j \in J] = [p_{k_j}(x) e^{i\lambda_j x} \,|\, j \in J],$$

where p_{k_j} is a polynomial of degree k_j.

Let F be a Blaschke product according to the standard definition. We will show that l ($\sim F$) has a basis of exponentials. Let g be an arbitrary element of $l^{\perp}$, and $Fh = \hat{g}$. If F has a zero of order m_j at $z = -\bar{\lambda}_j$, then it follows from the equation

$$(1.1) \qquad (Fh)^{k_j}(-\bar{\lambda}_j) = \int_0^{\infty} g(x)(ix)^{k_j} e^{-i\bar{\lambda}_j x}\, dx,$$

that $l_e (= [x^{m_j - 1} e^{i\lambda_j x} | j \in J]) \subseteq l$.

We need only show that $l \subseteq l_e$. If we substitute F_e ($\sim l_e$) for F in equation (1.1), and let g be an arbitrary element of $l_e^{\perp}$, we see that F has a zero of order m_j at $z = -\bar{\lambda}_j$. It then follows that F_e/F is an inner factor. Thus F_e is of the form $F_e = FF_c$, so that by Theorem B, $l \subseteq l_e$.

Conversely, we let $l = [x^{m_j-1}e^{i\lambda_j x} \mid j \in J]$, and show that $F\ (\sim l)$ is a standard Blaschke product. It follows from Equation (1.1) that F has a zero of order m_j at $z = -\bar{\lambda}_j$. Let F_e be the standard Blaschke product with zeros of order m_j at $Z = -\bar{\lambda}_j$. Then F/F_e is an inner factor, so that $F = F_e F_a$. According to Theorem B, $l_e \subseteq l$. From equation (1.1) we see that $l \subseteq l_e$, so that $l_e = l$. This implies that F_a is constant, and completes the proof.

DEFINITION. An inner factor $F(z)$ is said to be nonsingular if det $F(z)$ has no zeros (in the upper half plane).

The following results are well known, but we sketch the proof of Theorem D for the sake of completion:

THEOREM C. *Every scalar inner factor F can be factored in the form $F = F_e F_n$, where F_e is a Blaschke product and F_n is nonsingular. F_e and F_n are uniquely determined to within a constant unitary factor.*

THEOREM D. *If F is a scalar Blaschke product, and $F = F_a F_b$ then F_a and F_b are Blaschke products.*

PROOF. We factor F_a and F_b as in Theorem C to obtain $F_a = F_1 F_2$, $F_b = F_3 F_4$ (where F_1 and F_3 are Blaschke products and F_2 and F_4 are nonsingular). We must show that F_2 and F_4 are constant. We have $F = (F_1 F_3)(F_2 F_4)$. From Theorem C we see that $F_2 F_4$ is a constant unit. It then follows from the properties of inner factors that F_2 and F_4 are constant units.

2. **Results.** We consider the case where $V = W$ and is finite dimensional, and we prove the following two theorems:

THEOREM 1. *Every inner factor F can be factored in the form $F = F_e F_n$ where F_e is a Blaschke product and F_n is nonsingular.*

THEOREM 2. *F is a Blaschke product if and only if* det *F is a Blaschke product.*

We now have a generalization of Theorem D in the

COROLLARY. *If F is a Blaschke product and $F = F_a F_b$ where F_a and F_b are inner factors, then F_a and F_b are Blaschke products.*

To prove these results we need the following three lemmas; their proofs will be given in §3.

LEMMA 1. *If* det *F is constant, then F is a constant unitary operator.*

LEMMA 2. *Let l_λ be the span of the exponentials (with exponent λ) which are contained in l, and let $F\sim l$. Then* dim l_λ *equals the order of the zero of* det $F(z)$ *at* $z=-\bar{\lambda}$.

LEMMA 3. *Let $\beta(z)$ be the Blaschke product factor of* det F. *Let $l\sim F$, $l_\beta\sim\beta(z)I$, and let l_e be the span of the exponentials which are contained in l. Then $l_e\subseteq l_\beta$.*

PROOF OF THEOREM 1. Let l_e be the span of the exponentials contained in l $(\sim F)$, and let $F_e\sim l_e$. According to Theorem B, there exists an inner factor F_n such that $F=F_eF_n$. Taking determinants of both sides, we have $\det F=\det F_e \det F_n$. Let d, d_e, and d_n denote the order of the zero at $z=-\bar{\lambda}$ of det F, det F_e, and det F_n respectively, so that $d=d_e+d_n$. We must show that $d_n=0$. Since l and l_e have the same number of linearly independent exponentials with exponent λ, it follows from Lemma 2 that $d=d_e$.

PROOF OF THEOREM 2. Assume that det F is a Blaschke product. We factor F as in Theorem 1: $F=F_eF_n$. We must show that F_n is constant. Taking determinants, we have $\det F=\det F_e \det F_n$. By virtue of Theorem D, det F_n is a Blaschke product. But since according to Theorem 1, det F_n has no zeros, it follows that det F_n is constant. Then by virtue of Lemma 1, F_n is constant.

Conversely, assume that F is a Blaschke product. Let $\beta(z)$ be the Blaschke product factor of det F. Let $l\sim F$ and $l_\beta\sim\beta(z)I$. According to Lemma 3, $l\subseteq l_\beta$. Then by Theorem B, there exists an inner factor F_a such that $\beta(z)I=FF_a$. Taking determinants, we have $\beta^n(z)=\det F \det F_a$ (where $n=\dim V$). By virtue of Theorem D, det F is a Blaschke product.

3. Proofs of lemmas.

PROOF OF LEMMA 1. Since det $F(z)$ $(=c)$ is an inner factor, we have $\det(F^*(z)F(z))=\bar{c}c=1$. Let $\{a_j(z)\,|\,j=1,\cdots,n\}$ be the eigenvalues of $F^*(z)F(z)$. Since $\|F^*(z)F(z)\|\leqq 1$ (for $\operatorname{Im} z>0$), and $F^*(z)F(z)$ is nonnegative, we have $0\leqq a_j(z)\leqq 1$ $(j=1,\cdots,n)$. It then follows from the equation $\det(F^*(z)F(z))=\prod_{j=1}^n a_j(z)=1$ that all the eigenvalues of $F^*(z)F(z)$ are equal to one. Since $F^*(z)F(z)$ is symmetric, we thus have $F^*(z)F(z)=I$. Similarly $F(z)F^*(z)=I$, so that $F^*(z)=F^{-1}(z)$. Since $F(z)$ is analytic and $F^{-1}(z)$ is continuous, $F^{-1}(z)$ is analytic. But if $F(z)$ and $F^*(z)$ $(=F^{-1}(z))$ are both analytic, it follows that $F(z)$ is constant.

The proof of Lemma 2 is based on four sublemmas:

LEMMA 3.1. *Let $l=[ue^{i\lambda x}]$ where u is a given vector in V and* $\operatorname{Im}>0$. *Then F $(\sim l)$ is (up to a constant unitary operator on the right) of the form*

$$F = P + b(z)Q, \tag{3.1}$$

$$b(z) = \frac{z + \bar{\lambda}}{z + \lambda}, \tag{3.2}$$

Q is the orthogonal projection onto the one dimensional space spanned by u, and $P = I - Q$.

PROOF. Let $l_p \sim P + b(z)Q$. Then

$$((P + b(z)Q)h_v, u) = (h_v, P^*u) + b(z)(h_v, Q^*u) = b(z)(h_v, u).$$

Since $b(-\bar{\lambda}) = 0$, it follows from the equation

$$0 = ((P + b(-\bar{\lambda})Q)h_v(-\bar{\lambda}), u) = \int_0^\infty (a_v, u)e^{-i\bar{\lambda}x}\, dx$$

(where $a_v \in l_p^\perp$) that $l\ (= [ue^{i\lambda x}]) \in l_p$. From Theorem B we obtain $P + b(z)Q = FF_a$ (where $F \sim l$). Taking determinants of both sides of the factorization equation we have $b(z) = \det F \cdot \det F_a$. It follows from Theorem D that either $\det F$ or $\det F_a$ is constant. If $\det F$ were constant, then by Lemma 1, F would be a constant unitary operator. Clearly, this would imply that $l = \{0\}$. Thus by contradiction we see that $\det F_a$, and therefore F_a, must be constant, so that $l = l_p$.

DEFINITION. An inner factor of the form (3.1) is called a prime inner factor (at λ).

LEMMA 3.2. *Let F be an inner factor, and let* det *F have a zero of order at least m at* $z = -\bar{\lambda}$. *Then F can be factored in the form* $F = (\prod_{j=1}^m F_j)F_a$ *for some inner factor* F_a, *where* $\{F_j \mid j = 1, \cdots, m\}$ *are prime inner factors (at* λ*).*

PROOF. We use induction. Take $m = 1$. Since $\det F(-\bar{\lambda}) = 0$, there exists a vector u ($u \neq 0$) such that $F^*(-\bar{\lambda})u = 0$. Let $l \sim F$. It follows from the equation

$$0 = (h_v(-\bar{\lambda}), F^*(-\bar{\lambda})u) = (F(-\bar{\lambda})h_v(-\bar{\lambda}), u) = \int_0^\infty (a_v, u)e^{-i\bar{\lambda}x}\, dx$$

(where $a_v \in l^\perp$) that $ue^{i\lambda x} \in l$. Let $l_1 = [ue^{i\lambda x}]$, so that $l_1 \subseteq l$. Then by Theorem B, we have $F = F_1 F_a$, where $F_1 \sim l_1$. By virtue of Lemma 3.1, F_1 is a prime inner factor (at λ).

We now assume that the lemma is true for $m = k$, and consider the case $m = k + 1$. By our assumption we have $F = (\prod_{j=1}^k F_j)F_a$. Taking determinants, we have $\det F = b^k(z) \cdot \det F_a$. Since the order of the zero of $b^k(z)$ at $z = -\bar{\lambda}$ is k, we must have $\det F_a(-\bar{\lambda}) = 0$. Then, as shown above, $F_a = F_{k+1}F_b$ for some inner factor F_b, where F_{k+1} is a prime inner factor (at λ).

LEMMA 3.3. *The space* $l = [ux^{m-1}e^{i\lambda x} | u \in V]$ *corresponds to the inner factor* $F_b = b^m(z)I$.

Let $l_1 = [ux^{m-1}e^{i\lambda x} | u \in V]$. It follows from the equation

$$(3.3) \qquad (F(z)h_v, u)^{(k)} = \int_0^\infty (a_v, u)(ix)^k e^{izx}\,dx,$$

for $a_v \in l_1^\perp$, and $F = F_1$ $(\sim l_1)$, that F_1 has a zero of order at least m at $z = -\bar{\lambda}$. That is, $F_1^{(k)}(-\bar{\lambda}) = 0$, $k = 0, 1, \cdots, m-1$. Then we can factor F_1 in the form $F_1 = b^m(z)I \cdot F_c$, where F_c is an inner factor. We need only show that F_c is constant. Let $l_b \sim F_b = b^m(z)I$. It is easy to see from equation (3.3), for $\gamma_v = l_b^\perp$, and $F = F_b$, that $l_1 \subseteq l_b$. Then by Theorem B we have $b^m(z)I = (b^m(z)I \cdot F_c)F_a$. It follows that F_c is a constant unitary operator.

LEMMA 3.4. l $(\sim F)$ *consists of exponentials with exponent* λ *and is finite dimensional if and only if* $\det F = cb^m(z)$. *Also, if* $\det F = cb^m(z)$, *then* $\dim l = m$.

Let l consist of exponentials with exponent λ, and let $q-1$ be the order of the highest order exponential contained in l. Let $l_b = [ux^{q-1}e^{i\lambda x} | u \in V]$. Then $l \subseteq l_b$. According to Lemma 3.3, $b^q(z)I \sim l_b$. By virtue of Theorem B, we have $b^q(z)I = FF_a$ (where $F \sim l$). Taking determinants, we obtain $b^{qn}(z) = \det F \cdot \det F_a$ (where $n = \dim V$). It follows from Theorem D that $\det F$ is some power of $cb(z)$.

Conversely, let $\det F = cb^m(z)$. According to Lemma 3.2 we can factor F in the form

$$(3.4) \qquad F = \left(\prod_{j=1}^m F_j\right) F_a,$$

where F_j is the prime inner factor $P_j + b(z)Q_j$. By taking determinants of both sides of equation (3.4), we see that $\det F_a = c$. Then by virtue of Lemma 1, F_a is a constant unitary operator. Let $E_j = Q_j + b(z)P_j$ $(j = 1, \cdots, m)$. Clearly each E_j is an inner factor. Since

$$\left(\prod_{j=1}^m F_j\right) F_a F_a^* \left(\prod_{j=0}^{m-1} E_{m-j}\right) = b^m(z)I,$$

it follows from Theorem B that $l \subseteq l_b$. It then follows from Lemma 3.3 that l consists of exponentials with exponent λ.

Again, we assume that det $F = cb^m(z)$. We will show that dim $l = m$. As shown in the previous paragraph,

$$l\left(\sim \prod_{j=1}^{m} F_j\right) \subseteq l_b \ (\sim b^m(z)I).$$

According to Theorem B, we have $b^m(z)I = (\prod_{j=1}^m F_j)F_a$. Taking determinants, we have $b^{mn}(z) = b^m(z)\cdot\det F_a(z)$, so that $\det F_a(z) = b^{m(n-1)}(z)$. By virtue of Lemma 3.2, F_a can be factored in the form $F_a = \prod_{j=m+1}^{mn} F_j$. Let $E_k = \prod_{j=1}^{k} F_j$ $(k \leqq mn)$, and let $l_k \sim E_k$ (so that $l_m = l$, and $l_{mn} = l_b$). From Theorem B, we have $l_{k-1} \subseteq l_k$. Since no F_j is constant, we have $l_{k-1} \subset l_k$. By use of induction, we see that dim $l_k \geqq k$, and also that dim $l_{p+k} \geqq$ dim $l_p + k$, $(k+p \leqq mn)$. If dim $l_k > k$ for some fixed k, then dim $l_{mn} \geqq$ dim $l_k + mn - k > mn$. But it follows from Lemma 3.3 that dim $l_b = mn$. Thus, by contradiction, dim $l_k = k$, $k = 1, \cdots, mn$.

PROOF OF LEMMA 2. Let l contain (exactly) m linearly independent exponentials with exponent λ, and let l_λ be the span of these exponentials. From Theorem B, we have $F = F_\lambda F_a$. Taking determinants of both sides, we have det F = det $F_\lambda \cdot$ det F_a. According to Lemma 3.4, det $F_\lambda = cb^m(z)$, so that det F has a zero of order at least m at $z = -\bar{\lambda}$.

Conversely, we assume that det F has a zero of order m at $z = -\bar{\lambda}$. According to Lemma 3.2, F can be factored in the form $F = (\prod_{j=1}^m F_j)F_a$ for prime F_j's. Let $l_m \sim \prod_{j=1}^m F_j$. Since det $\prod_{j=1}^m F_j = b^m(z)$, it follows from Lemma 3.4 that l_m contains m linearly independent exponentials with exponent λ. Then by Theorem B, we have $l_m \subseteq l$, which completes the proof.

PROOF OF LEMMA 3. Let l contain an exponential $p(x)e^{i\lambda x}$ of order m. Since $(T_s - e^{i\lambda s})p(x)e^{i\lambda x}$ $(\in l)$ is an exponential of order $m-1$, we see that l contains at least $m+1$ linearly independent exponentials with exponent λ. By virtue of Lemma 2, $\beta(z)$ has a zero of order at least $m+1$, so that

$$(\beta(-\bar{\lambda})I\cdot h_v(-\bar{\lambda}), u)^{(k)} = 0, \quad \text{for } k = 0, \cdots, m.$$

It then follows from equation (3.3) that all exponentials with exponent λ of order m or less are contained in l_β.

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Courant Institute of Mathematical Sciences, New York University