LEFT ZERO SIMPLICITY IN SEMIRINGS

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A *topological semiring* is a Hausdorff space S together with two continuous associative operations on S such that one (called multiplication) distributes across the other (called addition). That is, $x(y+z)=xy+xz$ and $(x+y)z=xz+yz$ for all x, y, and z in S. Note that, in contrast to the purely algebraic situation [**1**], [**2**], we do not postulate the existence of an additive identity which is a multiplicative zero.

In this note we characterize compact additively commutative semirings which are multiplicatively left zero simple. This is accomplished by first examining semirings which are multiplicatively groups with zero and then proceeding to the general situation. For other remarks on compact semirings the reader is referred to [**3**], [**4**]. The notation follows closely that of topological semigroups [**5**].

An additive (a multiplicative) *ideal* of a semiring S is a nonempty set J such that $S+J\cup J+S\subset J(SJ\cup JS\subset J)$. By a *group with zero* we mean a semigroup S having a zero 0 (i.e. $0x=x0=0$ for all x in S) such that $S\backslash\{0\}$ is a group. Here, of course, 0 is a maximal proper ideal. It is well known that, in compact semigroups, maximal proper ideals are open [**6**].

THEOREM I. *In a compact additively commutative semiring S which is multiplicatively a group with zero, one of the following holds*:

(a) $S+S=0$,

(b) *S is a finite field*,

(c) *S is the lattice $\{0, 1\}$*,

(d) $$x+y=\begin{cases}x, & x=y\\ 0, & x\neq y\end{cases}$$ *and S is finite.*

PROOF. Let $E[+]$ represent the set of additive idempotents, i.e., elements e such that $e+e=e$. Note that $S+S$, $E[+]$, and $0+S$ are multiplicative ideals so that each is either all of S or the single point 0. If $S+S=0$ then we have (a) so suppose $S+S=S$. This and $E[+]=\{0\}$ give that S is a skew field [**7**, p. 26]. However $\{0\}$ is a maximal proper multiplicative ideal. Thus $\{0\}$ is open and S is discrete. Therefore S is finite. Now a finite skew field is a field so we have (b). Thus assume $E[+]=S$. From compactness we have a minimal additive

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ideal $K[+]$ which is a group. Since it must also be idempotent, $K[+]$ is a single point, say z. Clearly $z+x=z$ for all x in S. Suppose $0+S=S$. This implies $z\neq 0$. Also $z^2=z(z+1)=z^2+z=z$ so $z=1$ (1 being the multiplicative identity). Select u and v in S such that $u\neq 0\neq v$. We have $u+v=u+(vu^{-1})u=(1+(vu^{-1}))u=(z+(vu^{-1}))u=zu$ $=1u=u$ and, in the same way, $u+v=v$ so that $v=u$. Thus S is the lattice $\{0, 1\}$ i.e. case (c). Finally we may assume $0+S=0$. Suppose there are x and y in S such that $x\neq 0\neq y$, $x+S\neq y+S$, and $(x+S)$ $\cap(y+S)\neq\{0\}$. Pick p in $(x+S)\cap(y+S)$ such that $p\neq 0$. Clearly $p+S\subset(x+S)\cap(y+S)$. Thus $p+S$ is properly contained in $x+S$ or $y+S$ since they are unequal. Both cases, however, lead to a contradiction (of Theorem 3.4 in [**5**]) since $p+S$ is compact and (for example) $xp^{-1}(p+S)=x+S$. Therefore $x+S=y+S$ or $(x+S)\cap(y+S)$ $=\{0\}$ for all x and y in S. If $r\in x+S$ and $r\neq 0$ then $r\in(x+S)\cap(r+S)$ so $x+S=r+S$. Thus $x\in r+S$ giving an h in S so that $x=r+h$. Also, since $r\in x+S$, there is a k in S such that $r=x+k$. Hence $x=h+r$ $=h+(r+r)=(h+r)+r=x+r=x+(x+k)=(x+x)+k=x+k=r$. Consequently $x+S=\{0, x\}$ for all x in S. Thus if $x+y\neq 0$ then $x+y=x$ and similarly $x+y=y$ so $x=y$. Therefore, for any x and y in S, $x+y=0$ if $x\neq y$ and $x+y=x$ if $x=y$.

We shall now show that $S\backslash\{0\}$ is discrete. To see this choose $x\neq 0$ and note that if $\{x\}$ is not open then there exists a net $\{x_\alpha\}\to x$ such that $x_\alpha\neq x$ for all α. Now $x_\alpha+x=0$ for all α. But, by continuity of addition, $\{x_\alpha+x\}\to x+x$ which is x. Thus $x=0$. This is a contradiction so $\{x\}$ is an open set for $x\neq 0$. On the other hand, $S\backslash\{0\}$ is a maximal multiplicative group and thus closed. Thus $\{0\}$ is open and S is discrete. Now since S is compact it is finite. This completes the proof.

With the aid of Theorem I we are able to make a more general assertion in preparation for which we shall mention a lemma and two examples. A semigroup S is said to be *left zero simple* if S has a zero element 0 and each left ideal of S is either S or 0. Recall that a *left ideal* of S is a nonempty subset L of S such that $SL\subset L$.

LEMMA. *If S is a left zero simple semigroup having more than two elements then $S\backslash\{0\}$ is a left simple semigroup.*

The proof of this lemma is straightforward and can be found in [**8**, p. 68]. Of course, if S is compact then so is $S\backslash\{0\}$, since $\{0\}$ is a maximal proper ideal.

EXAMPLE I. Let $(S, +)$ be a compact commutative idempotent semigroup with an isolated unit 0 (i.e. $0+x=x+0=x$ for all x in S). For each x and y in S such that $x\neq 0\neq y$, define $xy=x$. For each x

in S, define $x0=0x=0$. It is a simple matter to check that $(S, +, \cdot)$ is a semiring and we shall omit the argument. S is the semiring in which we are interested.

If A is a closed ideal of a compact semigroup S then we can define a closed congruence α by identifying the points of A. This induces a semigroup S/α which is also called S/A [7]. We define S/A in a similar way in case A is a closed multiplicative and additive ideal of a compact semiring S.

EXAMPLE II. Let $(E, +)$ be a compact commutative idempotent semigroup with isolated zero 0 (i.e. $0+x=x+0=0$ for all x in E). For each x and y in E such that $x\neq 0\neq y$, define $xy=x$. For each x in S, define $x0=0x=0$. Clearly $(E, +, \cdot)$ is a semiring, Let $(H, \cdot)$ be a finite group with zero 0. For each x and y in H, let $x+y=x$ if $x=y$ and $x+y=0$ if $x\neq y$. Clearly $(H, +, \cdot)$ is a semiring. Thus $E\times H$ is a semiring under coordinate-wise addition and multiplication. Also $E\times\{0\}\cup\{0\}\times H$ is a closed ideal under both addition and multiplication. Thus

$$\frac{E\times H}{E\times\{0\}\cup\{0\}\times H}$$

is a semiring and this is the example in which we are interested.

THEOREM II. *In a compact additively commutative semiring S which is multiplicatively left zero simple, one of the following holds*:

(a) $S+S=0$,
(b) *S is a finite field*,
(c) *S is as in Example* I,
(d) *S is as in Example* II.

PROOF. If $S+S=0$ we have part (a) so suppose $S+S\neq 0$. Now $S+S$ is a multiplicative ideal so $S+S=S$. Since the set of additive idempotents, $E[+]$, is a multiplicative ideal $E[+]=0$ or $E[+]=S$. Suppose $E[+]=0$. Then S is an additive group [5, Theorem 4.3] and thus, in view of the lemma, an integral domain. Because $\{0\}$ is a maximal ideal and thus open, S is finite. But a finite integral domain is a field and we have part (b).

Suppose $E[+]=S$. Now the minimal additive ideal $K[+]$ must be an idempotent group, i.e., $K[+]$ is a single point, say z. Of course z is an additive zero (i.e., $z+x=x+z=z$ for all x in S).

Suppose $z\neq 0$. By the lemma $S\backslash\{0\}$ is multiplicatively a simple semigroup and thus the union of multiplicative groups [5]. Hence z is in some multiplicative group. If e is the identity of this group, we have $ez=ze=z$ so that $zz=(z+e)z=zz+z=z$. Now, according to pages 98 and 99 of [**5**], if $S\backslash\{0\}$ contained a nontrivial multiplicative group then $z(S\backslash\{0\})$ would be isomorphic to it. Furthermore, zS is a subsemiring and $zS=z(S\backslash\{0\})\cup\{0\}$. Applying Theorem I, we see that zS can have only two points. Thus $z(S\backslash\{0\})$ is a single point. Therefore $S\backslash\{0\}$ is not only multiplicatively left simple but also it contains no nontrivial multiplicative subgroups. From Theorem 2.5 of [**5**], we see that $S\backslash\{0\}$ is multiplicatively idempotent. If $p, q\in S\backslash\{0\}$ then there is an r in $S\backslash\{0\}$ so that $rq=p$. Thus $pq=(rq)q=r(qq)$ $=rq=p$. Finally, for any x in S, we see that $0+x=x0+xz=x(0+z)$ $=xz=x$, i.e., 0 is an additive unit. Therefore S is as in Example I and we have (c).

Now suppose $z=0$. Let $E[\cdot]$ represent the collection of multiplicative idempotent and pick e and f in $E[\cdot]\backslash\{0\}$. Then $Sf=f$ so there is a g in S such that $e=gf$ and $ef=(gf)f=g(ff)=gf=f$. Also $(e+f)(e+f)$ $=ee+ef+fe+ff=e+e+f+f=e+f$ so $e+f\in E[\cdot]$. Thus we have shown that $E[\cdot]$ is a semiring having the properties of E in Example II. Select any element e_0 of $E[\cdot]\backslash\{0\}$. As in the preceding paragraph e_0S is a semiring which is multiplicatively a group with zero. According to Theorem I, e_0S enjoys the properties of H in Example II. Define:

$$\Phi\colon \frac{E[\cdot]\times e_0S}{E[\cdot]\times\{0\}\cup\{0\}\times e_0S}\to S$$

by $\Phi((f, e_0x))=fe_0x$. Notice that $(E[\cdot]\times e_0S)\backslash(E[\cdot]\times\{0\}\cup\{0\}\times e_0S)$ $=(E[\cdot]\backslash\{0\})\backslash(e_0S\times\{0\})$. Now, according to Theorem 1 of [**9**], Φ restricted to $(E[\cdot]\backslash\{0\})\times(e_0S\backslash\{0\})$ is a multiplicative isomorphism onto $S\backslash\{0\}$. On the other hand, $\Phi^{-1}(0)=E[\cdot]\times\{0\}\cup\{0\}\times e_0S$ which is open and closed. Thus Φ is a multiplicative isomorphism.

Choose any x and y in e_0S and e and f in $E[\cdot]$. Suppose $x\neq y$. Then $\Phi((e, x)+(f, y))=\Phi((e+f, x+y))=\Phi((e+f, 0))=0$. Also $\Phi((e, x)+\Phi((f, y))=ex+fy$. If neither e nor f is 0 we have $e(ex+fy)$ $=ex+efy=ex+ey=e(x+y)=e0=0$ so $ex+fy=0$. If either e or f is 0, clearly $ex+fy=0$. Thus $\Phi((e, x)+(f, y))=0=\Phi((e, x))+\Phi((f, y))$. In case $x=y$ we have $\Phi((e, x)+(f, y))=\Phi((e+f, x))=ex+fy=\Phi((e, x))$ $+\Phi((f, y))$. Thus Φ is an isomorphism and we have part (d). This concludes the proof.

As a trivial consequence of the above theorem we have the following:

COROLLARY. *If S is a compact connected additively commutative semiring which is multiplicatively left zero simple then $S+S=0$.*

References

1. S. Bourne, *The Jacobson radical of a semiring*, Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 163–170.

2. ———, *On multiplicative idempotents of a potent semiring*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 632–638.

3. J. Selden, *A note on compact semirings*, Proc. Amer. Math. Soc. **15** (1964), 882–886.

4. ———, *Simplicity in compact semirings*, Amer. Math. Monthly (to appear).

5. A. D. Wallace, *The structure of topological semigroups*, Bull. Amer. Math. Soc. **61** (1955), 95–112.

6. R. J. Koch and A. D. Wallace, *Maximal ideals in topological semigroups*, Duke Math. J. **21** (1954), 681–685.

7. A. D. Wallace, *Notes on mobs*, Vol. I, Tulane University, New Orleans, La.

8. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. I, Math. Surveys No. **7**, Amer. Math. Soc., Providence, R. I., 1962.

9. A. D. Wallace, *The Rees-Suschkewitsch theorem for compact simple semigroups*, Proc. Nat. Acad. Sci. U.S.A. **24** (1956), 430–432.

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