A DISCONJUGACY CONDITION FOR $y'''+a_2y''+a_1y'+a_0y=0$

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1. **Introduction.** An nth order homogeneous linear differential equation is said to be *disconjugate* on the interval I of the real numbers provided no nontrivial solution of the equation has more than $n-1$ zeros (counting multiplicity) in I. C. de la Vallée Poussin in 1929 [5] and Zeev Nehari in 1962 [4] have developed conditions under which a general nth order linear differential equation will be disconjugate. For the 3rd order equation $L[y]=y'''+a_2y''+a_1y'+a_0y=0$ these results are respectively the following:

(i) $L[y]=0$ is disconjugate on the interval I if each a_i is continuous and bounded on I and $1 \geqq Ah + Bh^2/2 + Ch^3/6$ where $A=\sup|a_2(x)|$, $B=\sup|a_1(x)|$ and $C=\sup|a_0(x)|$ on I and h is the length of I.

(ii) $L[y]=0$ is disconjugate on the compact interval $[a, b]$ provided each a_i is continuous on $[a, b]$ and

$$1 \geqq \frac{(b-a)^2}{8}\int_a^b |a_0(x)|\,dx + \frac{b-a}{4}\int_a^b |a_1(x)|\,dx + \frac{1}{2}\int_a^b |a_2(x)|\,dx. \tag{1}$$

In 1963 Lasota [3] proved that $L[y]=0$ is disconjugate on the compact interval $[a, b]$ provided each a_i is continuous on $[a, b]$ and $1\geqq Ah/4+Bh^2/\pi^2+Ch^3/2\pi^2$. If $a_1(x)\leqq 0$ on $[a, b]$,

$$1 \geqq Ah/4 + Ch^3/2\pi^2 \tag{2}$$

together with the continuity of each a_i is sufficient to insure disconjugacy of $L[y]=0$ on $[a, b]$. The principal result of this paper is

THEOREM 2. *$L[y]=0$ is disconjugate on $[a, b]$ provided $a_1(x)\leqq 0$ on $[a, b]$, each a_i is continuous on $[a, b]$ and*

$$(2h+1)C[\exp(2hA) - \exp(hA) - hA]/A^2 \leqq 1 \tag{3}$$

where $2h=b-a$, $A=\max|a_2(x)|\neq 0$ and $C=\max|a_0(x)|$ over $[a, b]$.

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The differential equation $y'''+x^2y''+x^7y'+(x+1)y/600=0$ satisfies the conditions of Theorem 2 on $[-2, 0]$ and is therefore disconjugate on $[-2, 0]$; however neither (1) nor (2) are satisfied for this equation on $[-2, 0]$.

2. **An existence theorem.** The following existence theorem will be used extensively in the proof of Theorem 2:

THEOREM 1. *Let the function f which maps the strip*

$$S = \{(x, r, s, t): a \leqq x \leqq b, \ |r| + |s| + |t| < +\infty\}$$

into the reals be continuous on S and satisfy

(i) *$f(x, r, s, t)$ is nondecreasing in s for fixed x, r and t;*

(ii) *given a compact subset T of S there is a constant K^* depending on T such that for (x, r, s, t_1) and (x, r, s, t_2) in T $|f(x, r, s, t_1)-f(x, r, s, t_2)| \leqq K^*|t_1-t_2|$;*

(iii) *given a positive constant M there is another positive constant K depending on M such that $|f(x, r, 0, t)-f(x, r, 0, 0)| \leqq K|t|$ for $|r| \leqq M$, $|t| <+\infty$ and x in $[a, b]$.*

Then given a constant $M>0$ the BVP (*Boundary Value Problem*)

$$y''' = f(x, y, y', y''), \qquad y(a) = y'(a) = y'(b) = 0 \tag{4}$$

has a solution for $0<b-a\leqq 2\delta$ whenever

$$(2\delta + 1)K'[\exp(2\delta K) - \exp(\delta K) - \delta K]/K^2 \leqq M \tag{5}$$

where $K' = \sup\{|f(x, r, 0, 0)| : a\leqq x\leqq b, |r| \leqq M\}$.

PROOF. Let $M>0$ be given and let $s_1(x)=K^{-2}K'(\exp[K(b-x)] -\exp[K(b-a)])+K'(x-a)/K$. Then $s_1(a)=0$, $s_1'(x)\leqq 0$ and $s_1'' = -Ks_1'+K'$ on $[a, b]$. Also $s_1''(x)=K|s_1'(x)|+K'\geqq f(x, r, 0, s_1'(x)) -f(x, r, 0, 0)+f(x, r, 0, 0)$ for any x in $[a, b]$ and any r with $|r| \leqq M$. Therefore $s_1''(x)\geqq f(x, r, s_1(x), s_1'(x))$ for x in $[a, b]$ and $|r| \leqq M$. Similarly if

$$s_2(x) = K^{-2}K'(\exp[K(x - a)] - \exp[K(b - a)]) - K'(x - b)/K,$$

$$S_1(x) = K^{-2}K'(\exp[K(b - a)] - \exp[K(b - x)]) - K'(x - a_1)/K$$

and

$$S_2(x) = K^{-2}K'(\exp[K(b - a)] - \exp[K(x - a)]) + K'(x - b)/K,$$

we get the following inequalities: $s_2''(x)\geqq f(x, r, s_2(x), s_2'(x))$, $S_1''(x) \leqq f(x, r, S_1(x), S_1'(x))$ and $S_2''(x)\leqq f(x, r, S_2(x), S_2'(x))$ for x in $[a, b]$ and $|r| \leqq M$.

Next let z be a continuous real valued function on $[a, b]$. Then the BVP

$$y''(x) = f(x, z(x), y(x), y'(x)), \qquad y(a) = 0 = y(b) \tag{6}$$

has a unique solution [1; Theorem 6.3]. For each such z let u_z be the solution to (6). By the inequalities in the previous paragraph whenever $|z(x)| \leqq M$ on $[a, b]$ s_1 and s_2 are subfunctions for (6) and S_1 and S_2 are superfunctions for (6) [1; Theorem 2.2]. Moreover the values of s_1, s_2, S_1 and S_2 at a and b are such that s_1 and s_2 are underfunctions and S_1 and S_2 are overfunctions for (6) [1; p. 1058]. Then $\max[s_1(x), s_2(x)]$ is an underfunction for (6) [2; Theorem 3], and therefore $u_z(x) \geqq \min(\max[s_1(x), s_2(x)]) = s_1([a+b]/2) = s_2([a+b]/2)$. Similarly $u_z(x) \leqq S_1([a+b]/2) = S_2([a+b]/2) = -s_1([a+b]/2)$. Therefore $|z(x)| \leqq M$ implies

$$|u_z(x)| \leqq K^{-2}K'(\exp[K(b-a)] - \exp[K(b-a)/2] - K(b-a)/2)$$

for all x in $[a, b]$.

Let $\|\cdot\|$ denote the sup norm on $C[a, b]$. We shall show that there is a constant $M^* > 0$ such that $\|u_z'\| \leqq M^*$ whenever $\|z\| \leqq M$. Let x_0 be some point in (a, b), and consider the cases (i) $u_z(x_0) \leqq 0$ and (ii) $u_z(x_0) > 0$. In (i) let

$$\begin{aligned} s_1^*(x) = {} & K^{-2}K'(\exp[K(b-x)] - \exp[K(b-x_0)]) \\ & + K'(x-x_0)/K + u_z(x_0). \end{aligned}$$

Since s_1^* is a subfunction, $s_1^*(x_0) = u_z(x_0)$ and $s_1^{*\prime}(x) \leqq 0$ for $x \geqq x_0$, it must be true that $u_z'(x_0) \geqq s_1^{*\prime}(x_0)$. Also if

$$\begin{aligned} s_2^*(x) = {} & K^{-2}K'(\exp[K(x-a)] - \exp[K(x_0-a)]) \\ & - K'(x-x_0)/K + u_z(x_0), \end{aligned}$$

then $u_z'(x_0) \leqq s_2^{*\prime}(x_0)$. Thus

$$|u_z'(x_0)| \leqq K^{-1}K'(\exp[K(b-a)] + 1) \tag{7}$$

in case (i). For case (ii) change S_1 and S_2 to S_1^* and S_2^* as s_1 and s_2 were changed in case (i). Then (7) can be shown to hold in this case also. By using one underfunction and one overfunction (7) can be shown to hold for $x = a$ and $x = b$ also.

Now let B^* be the compact convex subset of $C[a, b]$ consisting of all z which satisfy $\|z\| + H(z) \leqq M$ where

$$H(z) = \sup |[z(u) - z(v)]/(u - v)|$$

taken over all distinct u and v in $[a, b]$. Define the mapping F from B^* into $C[a, b]$ by $F(z) = w$ where $w(x) = \int_a^x u_z(t)dt$ for each x in $[a, b]$. Pick $b - a \leqq 2\delta$ and let $E(x) = \exp(2x) - \exp(x) - x$. Then $|w(x)| \leqq 2\delta K'K^{-2}E(\delta K)$ and $|[w(u) - w(v)]/(u-v)| \leqq K'K^{-2}E(\delta K)$. Thus w is B^* provided (5) holds.

Next we show that F is continuous on B^*. Suppose that F is not continuous at z_0 in B^*. Then there is an $\epsilon_0>0$ and a sequence $\{z_n\}$ in B^* such that $\|z_n-z_0\|<1/n$ but $\|F(z_n)-F(z_0)\|\geqq\epsilon_0$ for all $n>0$. Let $u_n''=f(x, z_n, u_n, u_n')$. Then the sequences $\{u_n\}$ and $\{u_n'\}$ are both uniformly bounded sequences. Therefore there is a u_0 in $C^2[a, b]$ and a sequence $r(n)$ of positive integers such that for $i=0, 1, 2$, $u_{r(n)}^{(i)}$ converges uniformly to $u_0^{(i)}$ and $u_0''=f(x, z_0, u_0, u_0')$. Let $w_n=F(z_{r(n)})$ and $w_0=F(z_0)$. Then $\|w_n-w_0\|\leqq(b-a)\|u_{r(n)}-u_0\|$ which approaches 0 as n approaches ∞. This is impossible, and hence F must be continuous on B^*. Then by the Schauder-Tychonoff fixed point theorem there is a point z in B^* such that $z=F(z)$. This z is a solution to (4).

If we were to let $F(z)=w$ where $w(x)=-\int_x^b u_z(t)dt$ in the above proof, we would get an existence theorem for a BVP of the type $y'''=f(x, y, y', y'')$, $y'(a)=y(b)=y'(b)=0$ under the same conditions as in Theorem 1.

COROLLARY. *If the conditions of Theorem* 1 *are unchanged except that* (5) *is replaced by* $(2\delta+1)\delta K'\exp(2\delta K)\leqq MK$, *the conclusion of Theorem* 1 *is still valid.*

PROOF. Simply note that $E(x)\leqq x\exp(2x)$ for any $x>0$.

3. **The disconjugacy condition.** The Cauchy function $K(x, s)$ for the equation $L[y]=0$ is defined as follows: for s in $[a, b]$, $y(x)\equiv K(x, s)$ is the solution to the IVP $L[y]=0$, $y(s)=y'(s)=0$, $y''(s)=1$.

LEMMA. *Let* a_0, a_1 *and* a_2 *be continuous on* $[a, b]$. *Then* $L[y]=0$ *is disconjugate on* $[a, b]$ *if and only if* $K(x, s)>0$ *for each* x *and* s *in* $[a, b]$ *with* $x\neq s$.

PROOF. Clearly if $L[y]=0$ is disconjugate on $[a, b]$, $K(x, s)$ is positive for $x\neq s$. Let y be a solution to $L[y]=0$ with zeros at $x_1<x_2<x_3$ in $[a, b]$. We shall show that $K(x, s)>0$ for $x\neq s$ implies $y\equiv 0$ in $[a, b]$. $y(x)=c_1u_1(x)+c_2u_2(x)$ for all x in $[a, b]$, $L[u_i]=0$ and $u_i^{(j)}(x_1)=\delta_{ij}$ (the Kronecker δ) for $i=1$ and 2 and $j=0$, 1 and 2. But $u_2(x)\equiv K(x, x_1)$, so $u_2(x)>0$ for $x\neq x_1$. Now $y(x_2)=y(x_3)=0$, so if $c_1\neq 0$, then $(d/dx)[y(x)/u_2(x)]=c_1[u_1'(x)u_2(x)-u_1(x)u_2'(x)]/[u_2(x)]^2$ has a zero, say x^*, in (x_2, x_3). Let $h(x)=u_1(x)u_2(x^*)-u_1(x^*)u_2(x)$. Then $L[h]=0$, $h(x_1)=h(x^*)=h'(x^*)=0$. But $h''(x^*)\neq 0$ implies $h(x)$ is a nonzero multiple of $K(x, x^*)$, and hence $h(x_1)\neq 0$. Also $h''(x^*)=0$ implies $h(x)=0$ for all x in $[a, b]$, and then $u_1(x)=u_2(x)u_1(x^*)/u_2(x^*)$ which is impossible. This means that $c_1=0$, and since $y(x_2)=c_2u_2(x_2)=0$, $c_2=0$ also. Thus $y\equiv 0$ in $[a, b]$. It is clear that any nontrivial solution having a double zero in $[a, b]$ can have no other zero in $[a, b]$. Thus $L[y]=0$ is disconjugate on $[a, b]$.

THEOREM 2. *$L[y]=0$ is disconjugate on $[a, b]$ provided $a_1(x)\leqq 0$, each a_i is continuous on $[a, b]$ and*

$$(3) \qquad (2h+1)CE(hA)/A^2 \leqq 1$$

where $2h=b-a$, $A=\max a_2(x)\neq 0$ and $C=\max a_0(x)$ over $[a, b]$.

PROOF. We shall show that the Cauchy function $K(x, s)$ is positive for x and s in $[a, b]$ with $x\neq s$. Let g be some continuous function on $[a, b]$. Then given an $M>0$ by Theorem 1 the BVP

$$(8) \qquad L[y] = g, \qquad y(c) = y'(c) = y'(d) = 0$$

has a solution on the interval $[c, d]$ or $[d, c]$ for any $c\neq d$ in $[a, b]$ with $|c-d|\leqq 2\delta$ provided $(2\delta+1)(CM+\|g\|)E(\delta A)/A^2\leqq M$ or

$$(9) \qquad (2\delta+1)CE(\delta A)/A^2 \leqq 1 - \|g\|E(\delta A)(2\delta+1)/A^2M.$$

Let c and d satisfy $a\leqq c<d<b$, and let $d-c=2\delta$. Then $(2\delta+1)CE(\delta A)/A^2<1$ since $\delta<h$. It follows that if g is defined by

$$g(x) = \epsilon[a_0(x)(x-c)^3 + 3a_1(x)(x-c)^2 + 6a_2(x)(x-c) + 6]$$

and if $\epsilon>0$ is chosen sufficiently small, then (9) will hold. We conclude that (8) has a solution $u(x)$ on $[c, d]$ for this g. From the initial conditions it follows that

$$u(x) \equiv c_1K(x, c) + (x-c)^3.$$

Then $u'(d)=c_1K'(d, c)+3(d-c)^2=0$ implies that $K'(d, c)\neq 0$. Since $K''(c, c)=1$, it follows that $K(x, c)>0$ for $c<x\leqq b$. Similarly, if $a<c\leqq b$, then $K(x, c)>0$ for $a\leqq x<c$. It follows from the Lemma that $L[y]=0$ is disconjugate on $[a, b]$.

COROLLARY 1. *$L[y]=0$ is disconjugate on $[a, b]$ provided a_0, $a_1\leqq 0$ and a_2 are continuous on $[a, b]$ and $h(2h+1)C\exp(2hA)/A\leqq 1$ where $2h=b-a$.*

COROLLARY 2. *If $a_2=0$ on $[a, b]$, then $L[y]=0$ is discongugate on $[a, b]$ provided a_0 and $a_1\leqq 0$ are continuous on $[a, b]$ and $3h^2(2h+1)C\leqq 2$.*

PROOF. Simply observe that $\text{limit}_{x\to 0+} E(hx)/x^2=3h^2/2$.

It is known [2, Corollary of Theorem 7] that the second order differential equation $y''+py'+qy=0$ is disconjugate on $[a, b]$ provided p and q are continuous on $[a, b]$ and $q(x)\leqq 0$ on $[a, b]$. Thus

$y''' + a_2 y'' + a_1 y' = 0$ is disconjugate on $[a, b]$ provided a_2 and $a_1 \leqq 0$ are continuous on $[a, b]$. Theorem 2 gives a type of continuity condition for the disconjugacy of $L[y] = 0$ with respect to a_0 at $a_0 \equiv 0$. If the interval $[a, b]$ is fixed and a_2 and $a_1 \leqq 0$ are continuous on $[a, b]$, one can make the equation $L[y] = 0$ disconjugate on $[a, b]$ by choosing $\|a_0\|$ small enough.

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