ON POLYNOMIALS CHARACTERIZED BY A CERTAIN MEAN VALUE PROPERTY

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Let V denote the vector space of continuous real valued functions $f(x)$ satisfying the mean value property

$$f(x) = \frac{1}{N} \sum_{i=1}^{N} f(x + ty_i) \tag{1}$$

for $x \in R$, $0 < t < \epsilon_x$ (R denotes an n-dimensional region; x and y_i are abbreviations for $(x_1, \cdots, x_n)$, $(y_{i1}, \cdots, y_{in})$). We assume that the y_i's span E_n so that $1 \leqq n \leqq N$. We furthermore assume, without loss of generality, that $y_1, \cdots, y_n$ are linearly independent.

Friedman and Littman [5] have recently shown that V consists of polynomials of degrees $\leqq N(N-1)/2$. This bound is actually attained when the y_i's form the N vertices of an $(N-1)$-dimensional regular simplex [see 4, p. 264]. On the other hand it is known that for $n=2$, $\deg f \leqq N$ [see 4, Theorem 3.2]. The object of this paper is to obtain bounds on deg V and dim V, the bounds depending on N and n $(1 \leqq n \leqq N)$. We use the term deg V, to denote the maximum degree of the polynomials contained in V. We also characterize for fixed N and varying n $(1 \leqq n \leqq N)$ those configurations for which deg V and dim V attain their maximum.

THEOREM. *We have*

$$\deg V \leqq \sum_{j=1}^{n} (N - j), \qquad \dim V \leqq \prod_{j=0}^{n-1} (N - j) \tag{2}$$

so that for fixed N and varying n $(1 \leqq n \leqq N)$

$$\deg V \leqq \frac{N(N-1)}{2}, \qquad \dim V \leqq N! \quad . \tag{3}$$

The latter bounds are obtained if and only if

$$n = N \quad \text{or} \quad n = N - 1 \quad \text{and} \quad \sum_{i=1}^{N} y_i = 0.$$

REMARK. The bounds in (2) are not best possible. For instance, we have stated above that for $n=2$, deg $V \leqq N$ and this bound is best

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possible. For fixed n and N the problem of determining the maximum values of deg $\vee$, dim $\vee$ and the configurations for which these maximum values are attained remains open.

PROOF. We employ the following notation.

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right), \quad x \cdot y = x_1 y_1 + \cdots + x_n y_n,$$

$$P_k(x) = \sum_{i=1}^{N} (x \cdot y_i)^k \qquad (1 \leqq k < \infty).$$

It is shown in [5] that (1) is equivalent to the infinite system of homogeneous partial differential equations

$$P_k\left(\frac{\partial}{\partial x}\right) f = 0 \qquad (1 \leqq k < \infty) \tag{4}$$

and that $\vee$, which is thus the solution space of (4), is a finite dimensional space consisting of polynomials. Let R denote the ring of polynomials in $x_1, \cdots, x_n$ with real coefficients and let $\mathfrak{P}$ denote the ideal generated by the P_k's $(1 \leqq k < \infty)$. R, $\mathfrak{P}$, and $\vee$ are vector spaces over the reals and it is known that R is the direct sum of $\mathfrak{P}$ and $\vee$, i.e. $R = \mathfrak{P} \oplus \vee$ [see **2**, p. 53]. Thus the vector spaces $R/\mathfrak{P}$ and $\vee$ are isomorphic ($R/\mathfrak{P} \cong \vee$).

deg $\vee$ and dim $\vee$ will thus be determined if we know all the polynomials in $\mathfrak{P}$. We introduce the new variables $\xi_i = x \cdot y_i$ $(1 \leqq i \leqq N)$. Since the y_i's $(1 \leqq i \leqq N)$ are linearly independent we must have $\xi_{n+k} = \sum_{i=1}^{n} a_{ki} \xi_i$ $(1 \leqq k \leqq N-n)$ for an appropriate choice of real a_{ki}'s. Let R' denote the ring of polynomials in $\xi_1, \cdots, \xi_n$ with real coefficients and let $\mathfrak{P}'$ denote the ideal generated by the η_k's where $\eta_k = \sum_{i=1}^{n} \xi_i^k$ $(1 \leqq k \leqq \infty)$. We adopt the following notation:

$$\xi = (\xi_1, \cdots, \xi_n), \; i = (i_1, \cdots, i_n), \qquad \xi_i^{i} = \xi_1^{i_1}, \cdots, \xi_n^{i_n}.$$

It is known [see **1**, p. 41] that every polynomial $Q(\xi)$ can be expressed as

$$Q(\xi) = \sum{}' R_i \xi^i, \tag{5}$$

where the summation in $\sum'$ extends over those i's for which $0 \leqq i_j \leqq N-j$ $(1 \leqq j \leqq n)$ and R_i is a polynomial in $\eta_1, \cdots, \eta_N$. This representation is unique for $n = N$. Let c_i denote the constant term in R_i and let $S_i = R_i - c_i$. Clearly $S_i \in \mathfrak{P}'$. It follows from (5) that $Q(\xi) = \sum' C_i \xi^i + \sum' S_i \xi^i$ so that

$$Q(\xi) \equiv \sum{}' c_i \xi^i \pmod{\mathfrak{P}'}. \tag{6}$$

As there are $\prod_{j=0}^{n-1}(N-j)$ distinct ξ^i's, (6) shows that $\dim R'/\mathfrak{P}' \leqq \prod_{j=0}^{n-1}(N-j)$. Since $\vee \cong R/\mathfrak{P} \cong R'/\mathfrak{P}'$ we have $\dim \vee \leqq \prod_{j=0}^{n-1}(N-j)$. It follows furthermore from (6) that if Q is homogeneous and $\deg Q > \sum_{j=1}^{n}(N-j)$, then $Q \in \mathfrak{P}'$. This implies that if $P(x)$ is homogeneous and $\deg P > \sum_{j=1}^{n}(N-j)$ then $P \in \mathfrak{P}$. Thus $\deg \vee \leqq \sum_{j=1}^{n}(N-j)$.

If $n \leqq N-2$, then we conclude from (2) that $\deg \vee < N(N-1)/2$, $\dim \vee < N!$. It remains to treat the two cases: (a) $n=N$, (b) $n=N-1$. In case (a) the ξ^i's form a basis for $R'/\mathfrak{P}'$. For suppose that $\sum{}' c_i\xi^i \equiv 0 \pmod{\mathfrak{P}'}$ for some choice of real c_i's. Then $\sum{}' c_i\xi^i = \sum_{j=1}^{n} T_j(\xi)\eta_j$ where the T_j's are polynomials in $\xi_1, \cdots, \xi_n$. But each T_j has a representation (5). I.e. $T_j(\xi) = \sum{}' R_{ji}(\eta)\xi^i$ where the R_{ji}'s are polynomials in $\eta_1, \cdots, \eta_n$. Thus $\sum{}' c_i\xi^i = \sum_{j=1}^{n} \sum{}' R_{ji}\eta_i\xi^i = \sum{}' (\sum_{j=1}^{n} R_{ji}\eta_j)\xi^i$. Since the representation (5) is unique for $n=N$ we have

$$c_i = \sum_{j=1}^{n} R_{ji}\eta_j. \tag{7}$$

The left side of (7) is void of η's so that all R_{ji}'s and c_i's equal 0. Thus $\dim R'/\mathfrak{P}' = N!$ and since $\vee \cong R/\mathfrak{P} \cong R'/\mathfrak{P}'$, $\dim \vee = N!$. Now $\prod_{j=1}^{N-1} \xi_j^{N-j}$ has degree $N(N-1)/2$ and $\notin \mathfrak{P}'$. This implies that there exists a homogeneous polynomial of degree $N(N-1)/2 \notin \mathfrak{P}$. Hence $\deg \vee = N(N-1)/2$.

If $n=N-1$ then we distinguish two cases. If $\sum_{i=1}^{N} y_i \neq 0$, then it follows from [**1**, Theorem 2.2] that there exists an orthogonal transformation $x=Tx'$ such that $g(x')=f(Tx')$ is independent of x_n' and satisfies the equation

$$g(x_p') = \frac{1}{N}\sum_{i=1}^{N} g(x_p' + ty_{p,i}), \tag{8}$$

where $y_i = Ty_i'$, $x_p' = (x_1', \cdots, x_{n-1}')$, $y_{pi}' = (y_{i1}', \cdots, y_{i,n-1}')$. Let $\vee'$ denote the solution space of (8). Clearly $\deg \vee = \deg \vee'$, $\dim \vee = \dim \vee'$. It follows from (2) that $\deg \vee = \deg \vee' < N(N-1)/2$, $\dim \vee = \dim \vee' < N!$. If $\sum_{i=1}^{N} y_i = 0$, then define

$$\bar{x} = (x_1, \cdots, x_n, x_{n+1}),\ y_i = (y_{i1}, \cdots, y_{in}, 1)\ (1 \leqq i \leqq N),\ F(\bar{x}) = f(x).$$

We notice that $\sum_{i=1}^{N} \bar{y}_i \neq 0$. It therefore follows from [**3**, Theorem 2.2] that $\vee$ is identical with the solution space $\overline{\vee}$ of

$$F(\bar{x}) = \frac{1}{N}\sum_{i=1}^{N} F(\bar{x} + t\bar{y}_i). \tag{9}$$

Equation (9) is included in Case (a). It follows that $\deg \vee = \deg \overline{\nabla} = N(N-1)/2$, $\dim \vee = \dim \overline{\nabla} = N!$.

References

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