ELEMENTARY PROOF OF A THEOREM ON CONFORMAL RIGIDITY¹

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Let $f(z)$ be single valued and analytic in the annulus $A: a < |z| < b$, $f(A) \subset A$, and suppose that f maps every closed curve with winding number $+1$ about $z=0$ onto a curve with the same property. Then [1], [2], [3], [4] f is a rotation. Stimulated by a discussion with Professor A. Marden we shall give a short proof of this result with a minimum of topological notions.

By hypothesis, the integral of

$$\frac{1}{z} - \frac{f'(z)}{f(z)}$$

around any closed curve in A vanishes. Hence there exists a branch, $F(z)$, of $\log [z/f(z)]$ single valued in A . Let us define

$$u(re^{i\theta}) = \operatorname{Re} F(re^{i\theta}) = \log r - \log |f(re^{i\theta})|, \quad a < r < b,$$

$$I(r) = (2\pi)^{-1} \int_0^{2\pi} u(re^{i\theta}) d\theta.$$

By Cauchy's theorem applied to $z^{-1}F(z)$, $I(r)$ is independent of r . On the other hand, since $f(A) \subset A$,

$$(1) \quad \log(r/b) \leq u(re^{i\theta}) \leq \log(r/a), \quad a < r < b.$$

Therefore

$$\limsup_{r \rightarrow a} I(r) \leq 0, \quad \liminf_{r \rightarrow b} I(r) \geq 0.$$

Thus, $I(r) \equiv 0$, $a < r < b$. Hence, by (1),

$$\begin{aligned} J(r) &= (2\pi)^{-1} \int_0^{2\pi} |u(re^{i\theta})| d\theta = (2\pi)^{-1} \left[\int_{u \geq 0} u d\theta - \int_{u \leq 0} u d\theta \right] \\ (2) \quad &= -2(2\pi)^{-1} \int_{u < 0} u d\theta \leq 2 \log(b/r), \end{aligned}$$

or alternatively,

Received by the editors April 20, 1965.

¹ Work done with support from NSF grant GP 3904 with the University of Minnesota.

$$(3) \quad J(r) = 2(2\pi)^{-1} \int_{u>0} u \, d\theta \leq 2 \log(r/a), \quad a < r < b.$$

Since $|u(re^{i\theta})|$ is subharmonic, $J(r)$ is a convex function of $\log r$. Therefore $J(r) \leq \max[J(r_1), J(r_2)]$ whenever $a < r_1 \leq r \leq r_2 < b$. Letting $r_1 \rightarrow a$, $r_2 \rightarrow b$, and using (2), (3) we obtain

$$J(r) \equiv 0, \quad a < r < b.$$

Thus $u \equiv 0$, $|z^{-1}f(z)| \equiv 1$. Thus $f(z) = e^{ic}z$, c real.

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