IDEALS GENERATED BY PRODUCTS

A. P. HILLMAN, D. G. MEAD, K. B. O'KEEFE[1] AND E. S. O'KEEFE

In [1] Levi obtained results on the structure of the differential ideals $[y^n]$ and $[uv]$ and applied these results to the component theory of differential polynomials. The present paper uses Levi's methods to extend his main results on $[uv]$ to $[y_1 y_2 \cdots y_n]$. Since the y_i are independent indeterminates, $[y_1 \cdots y_n]$ is related to $[y^n]$ but is not quite a generalization. The results are motivated by and apply to differential algebra; however, we follow Levi's suggestion at the close of [1] in stating them in the more general form in which y_{ij} is not necessarily a derivative of y_i.

Let y_{ij} ($i=1, \cdots, n$ and $j=0, 1, \cdots$) be a set of independent indeterminates over a field F and let R be the polynomial ring in the y_{ij} over F. The signature of a monomial M in R is $D=(d_1, \cdots, d_n)$ if M has degree d_h in the y_{ij} with $i=h$. The weight of M is the sum of the j's for the factors y_{ij} of M. A polynomial of R is homogeneous with signature D if each of its terms has this signature; it is isobaric of weight w if each term has weight w.

For $j=0, 1, \cdots$ let x_j be a linear combination, with nonzero coefficients in F, of all the products $y_{1j_1} y_{2j_2} \cdots y_{nj_n}$ of weight j. Let I_t be the ideal $(x_0, x_1, \cdots, x_t)$ in R. Let $I=(x_0, x_1, \cdots)$ and let Q be the quotient ring of R modulo I. Below we describe functions $f(D)$ and $g(D)$ such that a monomial with signature D and weight w is in I_t if $w<f(D)$ and $t \geqq g(D)$ and such that for every D and $w \geqq f(D)$ there is a monomial with signature D and weight w that is not in I.

2. **Levi bases.** In the case $n=2$, Levi obtained the following bases for Q and R as vector spaces over F. Let $u_j=y_{1j}$ and $v_j=y_{2j}$. A product

$$P = u_{i_1} \cdots u_{i_r} v_{j_1} \cdots v_{j_s} \qquad (i_1 \leqq i_2 \leqq \cdots \leqq i_r,\ j_1 \leqq j_2 \leqq \cdots \leqq j_s) \tag{1}$$

of signature (r, s) is an α term if $s=0$ or $j_1 \geqq r$ and a β term otherwise. A λ term is of the form AX with A an α term and X a power product in the x_j. (X may be 1; thus the λ terms include Levi's α and γ terms.) Levi showed that the α terms are a basis for Q and the λ terms for R.

Let P' be a factor of P in (1) selected as follows. If P is an α term,

Received by the editors April 3, 1965.

[1] The work on this paper by one of the authors was supported by a grant from the American Association of University Women.

$P'=P$. If P is a β term, let $e=j_1+1$, $S=u_{i_e}v_{j_1}$, and $P'=P/S$. Let P_i be defined by $P_0=P$ and $P_{i+1}=P_i'$. Let $\theta(P)$ be the expression for P in the form $AS_1 \cdots S_a$ where a is the first i for which P_i is an α term, $A=P_a$, and $S_i=P_i/P_{i-1}$. The weight h_i of S_i satisfies $h_1 \leqq h_2 \leqq \cdots \leqq h_a$. Let $\lambda(P)=Ax_{h_1} \cdots x_{h_a}$. Levi showed that $P \rightarrow \lambda(P)$ is a 1-1 mapping of the set of all power products in the u_j and v_j onto the set of all λ terms.

3. **Extension to general** n. Let $P = Y_1 Y_2 \cdots Y_n$ where Y_k is a power product in the y_{ij} with $i=k$. For $2 \leqq k \leqq n$ we inductively define α_k terms in the $y_{1j}, \cdots, y_{kj}$ and an expression θ_k for $Y_1 \cdots Y_k$. When $k=2$, α_2 terms and $\theta_2=\theta(Y_1Y_2)$ are defined as in the previous section with y_{1j} in the role of u_j and y_{2j} in that of v_j. We assume the definition of α_m terms and of $Y_1 \cdots Y_m$ in the form $\theta_m=A_mS_{m1} \cdots S_{mb}$ where A_m is an α_m term and S_{mj} has signature $(1, \cdots, 1, 0, \cdots, 0)$, with m ones, and weight h_{mj} satisfying $h_{m1} \leqq h_{m2} \leqq \cdots \leqq h_{mb}$. Thinking of S_{mj} as u_j and $y_{m+1,j}$ as v_j, let $\theta(S_{m1} \cdots S_{mb} Y_{m+1}) = A^* S_{m+1,1} \cdots S_{m+1,c}$. We then define θ_{m+1} to be

$$A_{m+1}S_{m+1,1} \cdots S_{m+1,c} \tag{2}$$

where $A_{m+1}=A_mA^*$. The power product $Y_1 \cdots Y_{m+1}$ is defined to be an α_{m+1} term if c in (2) is zero, i.e., if $Y_1 \cdots Y_{m+1}=A_{m+1}$.

As before a λ term is of the form AX with A an α (i.e., α_n) term and X a power product in the x_j. Let $\theta(P)=\theta_n=AS_1 \cdots S_c$ with A an α term and S_j of weight h_j satisfying $h_1 \leqq \cdots \leqq h_c$ and let $\lambda(P)=Ax_{h_1} \cdots x_{h_c}$. The following outline of the process of showing that the α terms are a basis for Q and the λ terms for R is essentially as in Levi's work:

Since x_{h_1} is a linear combination of power products one of which is S_1, $(P/S_1)x_{h_1} \equiv 0 \pmod{I}$ can be solved as

$$P \equiv f_1N_1 + \cdots + f_sN_s \pmod{I} \tag{3}$$

where the f_i are in F and the N_i are power products in the y_{ij}. Any N in (3) which is not an α term can be replaced by an expression (3) in which N plays the role of P. Continuing the process, one ends in a finite number of steps with P congruent to a linear combination of α terms of the same signature and weight as P. Thus the α terms generate Q. It follows from the process of obtaining the θ_k and the 1-1 character of $P \rightarrow \lambda(P)$ in the case $n=2$ that, in the case of general n, $P \rightarrow \lambda(P)$ is a 1-1 mapping of the set of power products in the y_{ij} onto the set of λ terms. This establishes that the λ terms are a basis for R. Then the α terms are linearly independent modulo I and form a basis for Q.

The function $f(D)$ is the minimum of the weights of monomials with signature D that are not in I; hence $f(D)$ is the minimum of the weights of the α terms with signature D. An α term A is of the form $A_{n-1}A^*$ where A_{n-1} is an α_{n-1} term and A^* is an α_2 term in the $S_{n-1,j}$ and the y_{nj}. If $D=(d_1, \cdots, d_n)$ and A^* is of degree t in the S's, the weight of A is at least

$$f(d_1 - t, \cdots, d_{n-1} - t) + td_n \tag{4}$$

and is (4) if A_{n-1} is an α_{n-1} term of minimal weight for its signature and $Y_n=(y_{nt})^{d_n}$. Hence $f(D)$ is the minimum of (4) for $0\leqq t\leqq \min(d_1, \cdots, d_{n-1})$.

There are other ways of defining a basis for Q. Although the basis may be different, the function $f(D)$ does not depend on the basis. One such alternative shows that

$$\begin{aligned} f(D) = \min[&f(d_1-r, \cdots, d_a-r)+f(d_{a+1}-s, \cdots, d_b-s)+\cdots \\ &+f(d_{c+1}-t, \cdots, d_n-t)+f(r, s, \cdots, t)] \end{aligned} \tag{5}$$

where $r, s, \cdots, t$ range over all nonnegative integers such that the arguments in (5) are nonnegative.

It will be shown in another paper that $f(D)$ may be evaluated explicitly as follows. We assume without loss of generality that $d_1\leqq d_2 \leqq \cdots \leqq d_n$. For $2\leqq i\leqq n$ let $q_i=(d_1+\cdots+d_i)/(i-1)$ and let k be the smallest i for which q_i assumes its minimum. Let q and r be integers defined by $d_1+\cdots+d_k=(k-1)q+r$ and $0\leqq r<k-1$. Let $c_i=q-d_i$ for $i=1, \cdots, k$. Let $\sigma_1=c_1+\cdots+c_k$ and $\sigma_2=\sum_{i<j} c_ic_j$. Then $f(D)=\sigma_2+r\sigma_1+[(r+1)r/2]$.

It is easily seen from Levi's process that $g(d_1, d_2)$ may be chosen as d_1+d_2-2. This and the process described above show that $g(d_1, \cdots, d_n)$ may be chosen as

$$d_n + \min(d_1, \cdots, d_{n-1}) - 2.$$

Using the symmetry of the ideal I_t in the subscripts i of the y_{ij}, this may be improved to $g(D)=\min_{i\neq j}(d_i+d_j)-2$.

References

1. Howard Levi, *On the structure of differential polynomials and on their theory of ideals*, Trans. Amer. Math. Soc. **51** (1942), 532–568.

2. A. P. Hillman, D. G. Mead, K. B. O'Keefe and E. S. O'Keefe, *A dynamic programming generalization of xy to n variables*, Proc. Amer. Math. Soc. **17** (1966), 718–721.

University of Santa Clara and
University of Washington