THE DIFFERENTIAL IDEAL $[uv]$

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1. **Introduction.** Let $R\{u, v_l\}$ be the polynomial ring $R[u, u_1, u_2, \cdots : v_l, v_{l+1} \cdots]$ over R, a field of characteristic zero, with the derivation $D(y_i) = y_{i+1}$ for $y = u$ or v.

Let $\Omega = [uv_l]$ be the differential ideal generated by the form $X = uv_l$. Ω has the same elements as the ideal $(uv_l, (uv_l)_1, (uv_l)_2, \cdots)$, where the subscripts again denote derivatives.

A *power product* in $R\{u, v_l\}$ $P = u_{i(1)}u_{i(2)} \cdots u_{i(m)}v_{j(1)}v_{j(2)} \cdots v_{j(n)}$ is of *weight*, $\mathrm{w}(P) = \sum_{k=1}^{m} i(K) + \sum_{p=1}^{n} j(P)$, and *signature*, sig $(P) = \langle m, n\rangle$.

The following fundamental theorem is proved in [**3**].

Levi's Theorem. *If P is a power product in $R\{u, v\}$ and $\mathrm{w}(P) < m\cdot n$, then P is in the ideal $[uv]$.*

The purpose of this paper is to show that if P contains no proper factor which is in $[uv]$, and if $\mathrm{w}(P) \geqq mn$, then P is not in $[uv]$.

2. **Derivations and isomorphic images of $R\{u, v\}$.** Computations in $R\{u, v\}$ are simplified by working in an isomorphic image of $R\{u, v\}$, $R\{\bar{u}, \bar{v}\}$. $R\{\bar{u}, \bar{v}\}$ is the ring $R[\bar{u}, \bar{u}_1 \cdots, \bar{v}, \bar{v}_1 \cdots]$ with derivation $\overline{D}(\bar{y}_i) = \bar{y}_{i+1}$ for $y = u$ or v. The isomorphism is established by the mapping h: $h(\bar{u}_i) = u_i/i!$, $h(\bar{v}_j) = v_j/j!$. Thus $\overline{D}(\bar{u}_i)$ corresponds to $D(u_i)/(i+1)$ and $\overline{D}(\bar{v}_j)$ to $D(v_j)/j+1$. For typographical convenience, the bars will be omitted; hence $\overline{D}^n(\bar{u}\bar{v})$ is written $(uv)_n = \sum_{j=0}^{n} u_j v_{n-j}$.

Definition 2.1. $\overline{D}_1^l = D^l$ is defined on $R[u, u_1, \cdots, v_l, v_{l+n} \cdots]$ by

1. $D^l(u_i) = (i+1)u_{i+1} \qquad \text{for } i \geqq 0.$

2. $D^l(v_j) = \begin{cases} (j-l+1)v_{j+1} & \text{for } j \geqq l, \\ 0 & \text{for } j < l. \end{cases}$

3. If D_k^l has been defined, then $D_{k+1}^l = D^l(D_k)$.

Theorem 2.2. *Let h be the (nondifferential) isomorphism of $\mathfrak{R} = R[u, u_1, \cdots, v, v_1, \cdots]$ onto $\mathfrak{R}_l = R[u, u_1, \cdots, v_l, v_{l+1}, \cdots]$ determined by mapping u_i into u_i and v_j into v_{j+l}. Then*

$$h(D^0(P)) = D^l(h(P)). \tag{2.1}$$

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PROOF. It suffices to show (2.1) for $P=u_i$ and $P=v_j$. Suppose that $i\geqq 0$, then for $l\geqq 0$, $h(D^0(u_i))=h((i+1)u_{i+1})=(i+1)u_{i+1}=D^l(h(u_i))$; and for $j\geqq l$, $h(D^0(v_j)) = h((j+1)v_{j+1}) = (j+1)v_{j+l+1} = D^l(v_{j+l}) =D^l(h(v_j))$.

COROLLARY 2.3. *$\mathfrak{R}_l$ is closed under the operation D^l. Furthermore, the ideal $[uv_l]$, the image of $[uv]$ under the mapping h, is closed under D^l.*

COROLLARY 2.4. *Let $R\{u, v_l\}$ be the Ritt algebra $\langle\mathfrak{R}_l, D^l\rangle$, then $R\{u, v_l\}$ is isomorphic to $R\{u, v\}$.*

Let (uv) be the (algebraic) subring of $R\{u, v\}$ generated by uv; that is, (uv) is the set of elements of $R\{u, v\}$ divisible by uv.

THEOREM 2.5. *There is a module isomorphism g which maps $uR\{u, v\}/(uv)$ onto $R\{u, v_1\}$.*

PROOF. Let $I=(uv)$. If $a\in uR\{u, v\}/(uv)$, then for a unique b not involving v, $a=ub+I$. Define g by $g(a)=b$. Then $g(I)=0$, $g(c+I) =c/u$, if c does not contain v. Clearly if $r\in R$, $g(ra)=rg(a)$ and if a_1 and a_2 are elements of $uR\{u, v\}/(uv)$, then $g(a_1+a_2)=g(a_1)+g(a_2)$. Furthermore, for every c in $R\{u, v_1\}$, $g^{-1}(c)=uc+I$ and $g^{-1}(c)$ is an element of $uR\{u, v\}/(uv)$.

THEOREM 2.6. *Under g, $u[uv]/(uv)$ in $R\{u, v\}$ is mapped isomorphically on $[uv_1]$ in $R\{u, v_1\}$.*

PROOF. If $a\in u[uv]/(uv)$, then $a=uc+I$, where $c=\sum_0^m d(i)(uv)_i$ with $d(i)\in R\{u, v\}$. For $i>0$, $(uv)_i=(uv_1)_{i-1}+u_iv$; hence, $uc+I =u\sum_1^m d(i)(uv_1)_{i-1}+I$. Thus $g(a)=c$ and $c\in[uv_1]$. Further, $g^{-1}g(a) =a$. If any c is in $[uv_1]$, then $g^{-1}(c)=uc+I$, or $u\sum_0^m d(i)(uv_1)_i+I$. But then certain elements of I may be used to fill out the sums because $ud(i)u_iv\in I$ for every i. Therefore $u\sum_0^m d(i)(uv_1)_i+I =u\sum_1^{m+1} d(i-1)(uv)_i+I$, and g covers all of $[uv_1]$ and is an isomorphism.

COROLLARY 2.7. *If $Q\equiv 0[uv_1]$, then $u\cdot Q\equiv 0$ $[uv]$.*

PROOF. Using the g^{-1} of Theorem 2.2, $[uv_1]$ is mapped onto $u[uv]/(uv)$. Hence $uQ\equiv 0[uv]$ because $uQ\in uQ+I=g^{-1}(Q)$.

3. **The operator T_n.** Let $P=u_jUV$ be a power product of signature $\langle k, l\rangle$ and excess weight zero.

DEFINITION 3.1. T_n operates on V and is defined by

1. For $n=1$, $T_1(V)=D^1(V)-D^0(V)$.
2. If $T_{n-1}(V)$ has been defined, then

$T_n(V)=D^1(T_{n-1}(V))-T_{n-1}(D^0(V))$. (Note that T_n and D^l do not commute.)

THEOREM 3.2. *Let* $V = v_{j(1)}v_{j(2)} \cdots v_{j(l)}$, *then for* $n \leqq l$, $T_n(V) = (-1)^n n! \sum v_{j(1)} \cdots v_{t(1)+1} \cdots v_{t(n)+1} \cdots v_{j(l)}$, *with the summation extending over all products in which exactly* n *v-subscripts are raised by* 1. (*That is, no* $j_{(i)}$, $i = 1, \cdots, l$, *is raised more than* 1.) *If* $n > l$, $T_n(V) = 0$.

PROOF. The proof is by induction on n, keeping l fixed.

For $n = 1$,

$$\begin{aligned} T_1(V) &= D^1(V) - D^0(V) \\ &= \sum_{m=1}^{l} (j(m))v_{j(1)} \cdots v_{j(m)+1} \cdots v_{j(l)} \\ &\quad - \sum_{m=1}^{l} (j(m) + 1)v_{j(1)} \cdots v_{j(m)+1} \cdots v_{j(l)} \\ &= -\sum_{m=1}^{l} v_{j(1)} \cdots v_{j(m)+1} \cdots v_{j(l)}. \end{aligned}$$

For $n > 1$, assume that the theorem holds for values less than n. Let Z_n be the set of all functions z on $\{1, 2, \cdots, l\}$ to $\{0, 1\}$ with n occurrences of 1. The induction hypothesis may now be written, for $p < n$,

$$T_p(V) = (-1)^p p! \sum_{z \in Z_p} v_{j(1)+z(1)} \cdots v_{j(l)+z(l)}.$$

By definition $T_n(V) = D^1(T_{n-1}(V)) - T_{n-1}(D^0(V))$, and the induction hypothesis may be applied to T_{n-1}. Using the definition of D^0 and D^1, an expression for T_n may be derived as follows.

$$\begin{aligned} T_n(V) = {} & D^1\left((-1)^{n-1}(n-1)! \sum_{z \in Z(n-1)} v_{j(1)+z(1)} \cdots v_{j(l)+z(l)}\right) \\ & - T_{n-1}\left(\sum_{t=1}^{l} (j(t) + 1)v_{j(1)} \cdots v_{j(t)+1} \cdots v_{j(l)}\right) \\ = {} & (-1)^{n-1}(n-1)! \\ & \cdot \sum_{z \in Z(n-1)} \left(\sum_{t=1}^{l} (j(t) + z(t))v_{j(1)+z(1)} \cdots v_{j(t)+z(t)+1} \cdots v_{j(l)+z(l)}\right) \\ & - \sum_{t=1}^{l} (j(t) + 1)(-1)^{n-1}(n-1)! \\ & \cdot\left(\sum_{z \in Z(n-1)} v_{j(1)+z(1)} \cdots v_{j(t)+1+z(t)} \cdots v_{j(l)+z(l)}\right). \end{aligned}$$

These two sums are exactly comparable, the same t's and z's occurring in each. The sign of one term is + and the other −; the sum of the coefficients being

$$j(t) + z(t) - (j(t) + 1).$$

The sum coefficient is then -1 for exactly those terms where $z(t)=0$. It is 0 for the others. For $n \leqq l$, then, the terms unify, giving for each a new z, an element of Z_n; and for $n>l$, the terms cancel. In case $n \leqq l$, each element of Z_n can be found in n ways from as many elements of Z_{n-1}; hence, the new factor in the coefficient is $-n$. This concludes the proof.

The T-operator will now be applied to an arbitrary power product P of excess weight zero. First of all, if P contains any factor of negative excess weight, then P is in $[uv]$. Therefore, in particular, assume that P does not contain uv.

THEOREM 3.3. *Let* $P=u_1UV$, *then* $P \equiv uUT_1(V)[uv]$.

PROOF. Since P is a power product of excess weight zero, uUV has negative excess weight and is zero modulo $[uv]$ by Levi's Theorem. Mapping $\mathfrak{R}$ into itself by D^0, $uUV=0[uv]$ becomes

$$u_1UV + uD^0(U)V + uUD^0(V) \equiv 0\ [uv]. \tag{3.1}$$

Consider $Q=UV$ as a power product in $\mathfrak{R}_1$. Then $S=Uh^{-1}(V)$ in $\mathfrak{R}$ has signature $\langle k-1,\ l\rangle$ and weight $\mathrm{w}=kl-1-l<(k-1)l$; hence $S\equiv 0[uv]$. Under D^0, $S\equiv 0[uv]$ becomes

$$D^0(U)(h^{-1}(V)) + UD^0(h^{-1}(V)) \equiv 0\ [uv]. \tag{3.2}$$

Mapping $\mathfrak{R}$ into $\mathfrak{R}_1$, (3.2) becomes

$$D^0(U)V + UD^1(V) \equiv 0\ [uv_1]. \tag{3.3}$$

The derivation of $R\{u, v_1\}$, D', may be used in $[uv]$ because using the mapping g of Theorem 3.5, $g^{-1}D^1g$ maps $uR\{u,\ v\}/(uv)$ into itself and $u[uv]/(uv)$ into itself. Hence, by Corollary 2.7,

$$uD^0(U)V + uUD^1(V) \equiv 0\ [uv]. \tag{3.4}$$

Substituting (3.4) in (3.1) completes the proof.

LEMMA 3.4. *Let* $P=u_jUV$ *and let* h *map* $\mathfrak{R}$ *onto* $\mathfrak{R}_1$.
If $Q=Uh^{-1}(T_{j-1}(V))$, *then* $Q\equiv 0[uv]$.

PROOF. By Theorem 3.2, $\mathrm{w}(T_{j-1}(V^*))=\mathrm{w}(V)+(j-1)$ for each term $T_{j-1}(V^*)$ in $T_{j-1}(V)$. For each term Q^* in Q, $\mathrm{w}(Q^*)=\mathrm{w}(P)-j$

$+(j-1)-l=kl-l-1<(k-1)l$; and the signature of Q^* is $\langle k-1, l\rangle$. Hence $Q^*\equiv 0[uv]$ by Levi's Theorem.

THEOREM 3.5. *Let $P=u_jUV$, then for all $j>0$,*

$$P \equiv \frac{1}{j!} uUT_j(V) \ [uv]. \tag{3.5}$$

PROOF. The proof is by induction on j, and the case $j=1$ is covered by Theorem 3.3. Assume that (3.5) holds for values less than j. In $\mathfrak{R}$, $u_{j-1}UV\equiv 0[uv]$ by Levi's Theorem. Under D^0, we have

$$ju_jUV \equiv (-u_{j-1}D^0(U)V - u_{j-1}UD^0(V)) \ [uv]. \tag{3.6}$$

Applying the induction hypothesis to each term on the right (3.6) becomes

$$ju_jUV \equiv \left(-\frac{1}{(j-1)!} uD^0(U)T_{j-1}(V) - \frac{1}{(j-1)!} uUT_{j-1}(D^0(V))\right) [uv]. \tag{3.7}$$

Map $\mathfrak{R}$ onto $\mathfrak{R}_1$ by h and consider $Q=UT_{j-1}(V)$ as a power product in $\mathfrak{R}_1$. Then $S=Uh^{-1}(T_{j-1}(V))$ is in $[uv]$ by Lemma 3.4. Under D^0, $S\equiv 0[uv]$ becomes

$$D^0(U)h^{-1}(T_{j-1}(V)) + UD^0(h^{-1}(T_{j-1}(V))) \equiv 0 \ [uv]. \tag{3.8}$$

Mapping $\mathfrak{R}$ onto $\mathfrak{R}_1$, (3.8) becomes

$$D^0(U)T_{j-1}(V) + UD^1(T_{j-1}(V)) \equiv 0 \ [uv_1]. \tag{3.9}$$

By Corollary 2.7, we get

$$uD^0(U)T_{j-1}(V) + uUD^1(T_{j-1}(V)) \equiv 0 \ [uv]. \tag{3.10}$$

Substituting (3.10) in (3.7) completes the proof.

4. **The converse of H. Levi's Theorem for** $[uv]$. Let $P=u_{i(1)}u_{i(2)} \cdots u_{i(k)}v_{j(1)}v_{j(2)} \cdots v_{j(l)}$ be of signature $\langle k, l\rangle$ and weight w. Assume that P has no factor of negative excess weight. By Theorem III of [**4**], without loss of generality, we may set $\mathrm{w}(P)=kl$. If a sequence of k transformations exist such that

(4.1)
(1) $V=v_{j(1)} \cdots v_{j(l)}$ is changed to v_k^l,
(2) in the tth transformation exactly $i(t)$ v-subscripts are increased by one,
(3) $U=u_{i(1)} \cdots u_{i(k)}$ is changed to u^k;

then P may be written congruent to a linear combination of α-terms of the same weight and signature as P, [3]. P is of excess weight zero and thus $P \equiv c u^k v_k^l [uv]$. The only question concerns the coefficient c, which is not zero, but is $(-1)^{i_1+i_2+\cdots+i_k} m$ where m is the number of sequences which transform V to v_k^l. Thus $c=0$ if and only if $m=0$, and we have proved

THEOREM 4.1. *If $P=UV$ has a nonnegative weight matrix, then P is not in $[uv]$ if and only if V can be transformed to v_k^l by a sequence of n steps, in the tth of which exactly $i(t)$ v-subscripts are increased by one.*

It remains to characterize those U and V for which (4.1) exists.

At the tth step, suppose a power product M is transformed into a power product N as follows: u_t in M is replaced with u and the lowest t v-subscripts (assuming that $j(1) \leqq j(2) \leqq \cdots \leqq j(t) \leqq \cdots \leqq j(l)$) are increased by one. Now, if N contains a factor with negative excess weight, then the same is true of M. More generally, we prove

THEOREM 4.2. *Let M be a power product of signature $\langle k, l\rangle$ containing u_t, $t>0$ and t v's, $v_{j(1)} \cdots v_{j(t)}$, and let*

$$N = M \frac{u v_{j(1)+1} \cdots v_{j(t)+1}}{u_t v_{j(1)} \cdots v_{j(t)}}.$$

Then if G is any factor on N with excess weight $e(G)$, there is a factor F of M with excess weight $e(F) \leqq e(G)$.

PROOF. We may assume G has u as a factor; otherwise, by reducing the subscripts in G that have been raised we get a factor G^* of M with $e(G^*) \leqq e(G)$. Therefore G is of the form uU_1V, where U_1 is a factor of U; notationally, let $U_1=U$. If V involves no unchanged subscripts, then lowering the n subscripts of V that have been raised we get V^* and a factor UV^* of M with $e(UV^*)=\mathrm{w}(U)+\mathrm{w}(V)-n-(k-1)n=e(uUV)$. If V involves all the changed subscripts, then similarly $e(u_tUV^*)=t+\mathrm{w}(U)+(W(V)-t)-k \deg V^*=e(uUV)$. If V involves an unchanged subscript but not all changed ones, we can exchange an unchanged subscript for a changed one except in the case that all the changed subscripts of N are $j(t)+1$ and all the unchanged subscripts of G are $j(t)$. Thus a reduction is achieved except in the case that G is of the form $uUv^p_{j(t)+1}v^q_{j(t)}$, $p<t$, $q>0$. Consider the cases (1) $k \geqq j(t)+1$ and (2) $k \leqq j(t)$.

In case 1,

$$e(uUv^{p+1}_{j(t)+1}v^{q}_{j(t)}) \leqq e(uUv^{p}_{j(t)+1}v^{q}_{j(t)}).$$

In case 2,

$$e(uU\overset{p}{v}_{j(t)+1}\overset{q-1}{v}_{j(t)}) \leqq e(uU\overset{p}{v}_{j(t)+1}\overset{q}{v}_{j(t)}).$$

In either case, a factor F of M has been found such that $e(F) \leqq e(G)$, and the proof is complete.

COROLLARY 4.3. *If $P = UV$ has a nonnegative weight matrix and excess weight zero, then $P \not\equiv 0[uv]$.*

PROOF. By symmetry we may assume that $V \not\equiv 0(v)$. By Theorem 4.2, there is a sequence of transformations satisfying (4.1) which transforms P into the α-term $u^k v_k^l$.

COROLLARY 4.4. *If $P = u_i v_j$, the smallest exponent q such that $P^q \equiv 0[uv]$ is $q = i+j+1$.*

PROOF. $Q = (u_i v_j)^{i+j+1}$ has negative excess weight; hence, by Levi's Theorem is in $[uv]$. On the other hand, $S = (u_i v_j)^{i+j}$ has a nonnegative weight matrix, excess weight zero, and is not in $[uv]$ by Corollary 4.3. This solves Ritt's exponent problem for $[uv]$, ([**1**], p. 177).

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