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THE UNIVERSITY OF BRITISH COLUMBIA

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q-ANALOGUES OF CAUCHY'S FORMULAS

WALEED A. AL-SALAM

1. Let q be a given number and let α be real or complex. The αth "basic number" is defined by means of $[\alpha]=(1-q^\alpha)/(1-q)$. This has served as a basis for an extensive amount of literature in mathematics under such titles as Heine, basic, or q-series and functions. The basic numbers also occur naturally in many theta identities. The works of Jackson (for bibliography see [2]) and Hahn [3] have stimulated much interest in this field.

One important operation that is intimately connected with basic series as well as with difference and other functional equations is the q-derivative of a function f. This is defined by

$$Df(x) = \frac{f(qx)-f(x)}{x(q-1)}\,. \tag{1.1}$$

Jackson defined the operations, which he called q-integration,

$$\int_0^x f(t)d(q,t) = x(1-q)\sum_{k=0}^{\infty} q^k f(xq^k) \tag{1.2}$$

and

$$\int_x^\infty f(t)d(q,t) = x(1-q)\sum_{k=0}^{\infty} q^{-k} f(xq^{-k}) \tag{1.3}$$

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provided the series on the right hand side are convergent. Both (1.2) and (1.3) are inverse operations to (1.1) and are analogues of the definite integrals $\int_0^x f(t)dt$ and $\int_x^\infty f(t)dt$ respectively. In fact each approach the respective integral as $q\to 1$ when f is Riemann integrable in the intervals of integration.

The definite q-integral $\int_a^x f(t)d(q, t)$ is defined by means of

$$\int_a^x f(t)d(q,t) = \int_0^x f(t)d(q,t) - \int_0^a f(t)d(q,t). \tag{1.4}$$

The purpose of this note is to obtain q-analoques of Cauchy's formulas for multiple integrals

$$\begin{aligned}\int_a^x \int_a^{x_{n-1}} \cdots \int_a^{x_1} f(t)\, dt dx_1 \cdots dx_{n-1} \\ = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\, dt\end{aligned} \tag{1.5}$$

and

$$\begin{aligned}\int_x^\infty \int_{x_{n-}}^\infty \cdots \int_x^\infty f(t)\, dt dx_1 \cdots dx_{n-1} \\ = \frac{1}{(n-1)!}\int_x^\infty (t-x)^{n-1} f(t)\, dt.\end{aligned} \tag{1.6}$$

The q-analogues of (1.4) and (1.5) are given in Theorems 1 and 2 below. We remark that (3.1) and (3.2) can be regarded as a transformations of n-fold infinite series to a single infinite series. These are basically different from a finite analogue of (1.5) that has been recently given by Traub [4]. We further note that (3.1) and (3.2) can be used to define fractional q-integral and q-derivative in the same way that (1.4) and (1.5) have been used to define fractional integrals and derivative. This we shall give elsewhere.

2. Let for a nonnegative integer n, $[0]!=1$, $[n]!=[n][n-1]\cdots[1]$ if $n>0$, and let further, $0<q<1$ and

$$(a+b)_0 = 1,\ (a+b)_n = (a+b)(a+bq)\cdots(a+bq^{n-1}) \text{ if } n > 0.$$

We shall also use the notation

$$(a)_0 = 1, \quad (a)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}) \quad \text{if } n>0,$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k (q)_{n-k}},$$

We first give the following lemmas:

LEMMA 1. *For integers* $n \geqq 1$ *and* $m \geqq 0$ *we have*

$$A_m = \sum_{j=0}^{m} q^j (q^j - q^{m+1})_{n-1} = \frac{(q^{m+1})_n}{1 - q^n}. \tag{2.1}$$

PROOF. We have

$$\begin{aligned} \sum_{m=0}^{\infty} A_m u^m &= \sum_{m=0}^{\infty} u^m \sum_{j=0}^{\infty} q^j (q^j - q^{m+1})_{n-1} \\ &= \sum_{j=0}^{\infty} u^j q^{nj} \sum_{m=0}^{\infty} (q^{m+1})_{n-1} u^m \\ &= \frac{1}{1 - uq^n} \sum_{m=0}^{\infty} \frac{(q)_{m+n-1}}{(q)_m} u^m, \end{aligned}$$

so we have

$$\sum_{m=0}^{\infty} A_m u^m = \frac{(q)_{n-1}}{1 - uq^n} \sum_{m=0}^{\infty} \frac{(q^n)_m}{(q)_m} u^m. \tag{2.2}$$

Let us recall the formula [1, p. 66]

$$\sum_{k=0}^{\infty} \frac{(a)_k}{(q)_k} u^k = \prod_{r=0}^{\infty} \frac{1 - auq^r}{1 - uq^r},$$

so that (2.2) can now be written as

$$\begin{aligned} \sum_{m=0}^{\infty} A_m u^m &= \frac{(q)_{n-1}}{1 - uq^n} \prod_{r=0}^{\infty} \frac{1 - uq^{r+n}}{1 - uq^r} \\ = (q)_{n-1} \prod_{r=0}^{\infty} \frac{1 - uq^{n+r+1}}{1 - uq^r} &= (q)_{n-1} \sum_{k=0}^{\infty} \frac{(q^{n+1})_k}{(q)_k} u^k. \end{aligned}$$

Equating coefficients of u^m we get (2.1).

LEMMA 2. *For integral* $m \geqq 1$ *we have*

$$\sum_{j=1}^{m} q^{-j} (q^{-m-1} - q^{-j})_{n-1} = \frac{q^{-n(m+1)+1}}{1 - q^n} (q^m)_n. \tag{2.3}$$

The proof of this lemma is similar to that of Lemma 1.

LEMMA 3. *For nonnegative integer n,*

$$(2.4)\qquad \begin{aligned} &a^{n+1}(1-q^n)\sum_{j=0}^{\infty} q^j(q^j-q^{k+1})_{n-1} \\ &\qquad - ax(1-q^n)\sum_{j=1}^{\infty} q^j(xq^j-aq^{k+1})_{n-1} - a^{n+1}(q^{k+1})_n \\ &= -a(x-aq^{k+1})_n. \end{aligned}$$

PROOF. To prove this lemma we recall the identity due to Euler

$$(2.5)\qquad (a+b)_n = \sum_{r=0}^{n} (-1)^r \begin{bmatrix} n \\ r \end{bmatrix} q^{r(r-1)/2} a^{n-r} b^r,$$

so that

$$\sum_{j=0}^{\infty} q^j(q^j-q^{k+1})_{n-1} = \sum_{r=0}^{n-1} (-1)^r \begin{bmatrix} n-1 \\ r \end{bmatrix} q^{r(r-1)/2} \frac{q^{r(k+1)}}{1-q^{n-r}}.$$

Hence we get

$$(2.6)\qquad \begin{aligned} &a^{n+1}(1-q^n)\sum_{j=0}^{\infty} q^j(q^j-q^{k+1})_{n-1} \\ &\qquad = a^{n+1}\sum_{r=0}^{n-1}(-1)^r \begin{bmatrix} n \\ r \end{bmatrix} q^{r(r-1)/2} q^{r(k+1)} \\ &\qquad = a^{n+1}(1-q^{k+1})_n - (-1)^n q^{n(n-1)/2+n(k+1)}. \end{aligned}$$

Similarly

$$(2.7)\qquad \begin{aligned} &ax(1-q^n)\sum_{j=0}^{\infty} q^j(xq^j-aq^{k+1})_{n-1} \\ &\qquad = a\left[(x-aq^{k+1})_n - (-1)^n q^{(1/2)n(n-1)+n(k+1)} a^n\right]. \end{aligned}$$

Substituting (2.6) and (2.7) in the left hand side of (2.4) we get the right hand side of (2.4).

3. We now prove our main results.

THEOREM 1. *If $n \geqq 1$ is a given integer*

$$(3.1)\qquad \begin{aligned} I^n f(x) &= \int_a^x \int_a^{x_{n-1}} \cdots \int_a^{x_1} f(t)d(q,t)d(q,x_1)\cdots d(q,x_{n-1}) \\ &= \frac{1}{[n-1]!}\int_a^x (x-qt)_{n-1} f(t)d(q,t), \end{aligned}$$

and

THEOREM 2. *If* $n \geqq 1$ *is an integer*

$$K^n f(x) = \int_x^\infty \int_{x_{n-1}}^\infty \cdots \int_{x_1}^\infty f(t) d(q, t) d(q, x_1) \cdots d(q, x_{n-1}) \tag{3.2}$$
$$= \frac{q^{-(1/2)n(n-1)}}{[n-1]!} \int_x^\infty (t - x)_{n-1} f(tq^{1-n}) d(q, t).$$

Both of these theorems can be proved by induction. We give the proof of the first as the proof of the second is similar. To prove Theorem 1 assume (3.1) is true for $n = N$. We have

$$I^{N+1} f(x) = \frac{1-q}{[N-1]!} \int_a^x \sum_{k=0}^\infty q^k \{ t(t - tq^{k+1})_{N-1} f(tq^k)$$
$$- a(t - aq^{k+1})_{N-1} f(aq^k) \} d(q, t)$$
$$= \frac{(1-q)^2}{[N-1]!} \sum_{k=0}^\infty \sum_{j=0}^\infty q^{k+j} (1 - q^{k+1})_{N-1} \{ x^{N+1} q^{Nj} f(xq^{k+j})$$
$$- a^{N+1} q^{Nj} f(aq^{k+j}) \}$$
$$- \frac{(1-q)^2}{[N-1]} \sum_{k=0}^\infty \sum_{j=0}^\infty q^{k+j} f(aq^k) \{ ax(xq^j - aq^{k+1})_{N-1}$$
$$- a^{N+1} (q^j - q^{k+1})_{N-1} \}.$$

This reduces after some simplification to

$$[N-1]! I^{N+1} f(x) = (1-q)^2 \sum_{s=0}^\infty q^s f(xq^s)(x^{N+1} - a^{N+1}) \sum_{j=0}^\infty q^j (q^j - q^{s+1})_{N-1}$$
$$- (1-q)^2 \sum_{s=(}^\infty q^s f(aq^s) ax \sum_{=0}^\infty q^j (xq^j - aq^{s+1})_{N-1}$$
$$+ (1-q^2) \sum_{s=0}^\sigma q^s f(aq^s) a^{N+1} \sum_{j=0}^\infty q^j (q^j - q^{s+1})_{N-1}.$$

Evaluating the inside sums in the right hand side of the above equation by means of Lemmas 1 and 3 we get

$$I^{N+1} f(x) = \frac{(1-q)}{[N]!} \sum_{s=0}^\infty q^s \{ x(x - xq^{s+1})_N f(xq^s) - a(x - aq^{s+1})_N f(aq^s) \}$$
$$= \frac{1}{[N]!} \int_a^x (x - qt)_N f(t) \, d(q, t).$$

Since (3.1) is true for $n=1$ our proof is complete. We remark that Lemma 2 is required in the proof of Theorem 2.

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University of Alberta at Calgary