SOME WILD IMBEDDINGS IN CODIMENSION TWO

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1. Statement of the Theorem.

THEOREM. *For every $n \geq 3$, the following exist in E^n :*

I. *A wild $(n-2)$ -sphere which is the union of two flat $(n-2)$ -cells whose common boundary is a flat $(n-3)$ -sphere.*

II. *A wild $(n-2)$ -cell which is the union of two flat $(n-2)$ -cells intersecting in a flat $(n-3)$ -cell common to their boundaries.*

III. *A wild $(n-2)$ -sphere which fails to be locally flat at precisely one point; moreover, it fails to be locally tame at that point.*

IV. *A wild $(n-2)$ -cell which fails to be locally flat at precisely one point; moreover, the point is an interior point and the cell fails to be locally tame there.*

The spheres in III were constructed by Cantrell and Edwards [2] as an example of a wild $(n-2)$ -sphere which is locally flat at all save possibly one point; they asked whether or not the spheres were locally flat at that point, answered here in the negative. Examples I and II give examples of 2-complexes wildly imbedded in E^4 but such that each simplex is tamely imbedded; Edwards has shown [3] that for a 2-complex imbedded in E^n for $n \geq 6$, this cannot occur; for $n = 5$, the question is open. Examples III and IV show that Theorems 3 and 6 of [3], proved for $n \geq 6$, fail for $n = 4$.

2. Definitions and lemmas. We regard E^k , $k \leq n$, as the subset of E^n defined by $x_i = 0$ for all $i > k$, and define the positive half k -space E_+^k to be the subset of E^k defined by $x_k \geq 0$.

A topological k -manifold M in E^n is said to be *locally flat* at a point $p \in M$ if there is a neighborhood U of p (in E^n) and a homeomorphism h carrying the pair $(U, U \cap M)$ onto the pair (E^n, E^k) if $p \notin \text{bdry } M$, or to (E^n, E_+^k) if $p \in \text{bdry } M$. M is said to be *locally tame* at $p \in M$ if there is a neighborhood U of p (in E^n) and a homeomorphism g of U onto E^n carrying $U \cap M$ onto a subpolyhedron of E^n . M is said to be *tame* if there is a homeomorphism of E^n onto itself carrying M onto a subpolyhedron of E^n . A cell (sphere) in E^n is said to be *flat* if there is

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a homeomorphism of E^n onto itself carrying the cell (sphere) onto a simplex (the boundary of a simplex).

The first lemma provides a precise way to prove that certain manifolds fail to be locally flat at a point (e.g., the cone over any nontrivial knot). It is a direct generalization of two theorems of Fox and Artin [5], the proof being essentially the same.

LEMMA 1. *If a k -manifold M in E^n , $k \leq n-2$, is locally flat at $p \in M$, then given any monotone sequence $V_1 \supset V_2 \supset \dots$ of closed neighborhoods of p (in E^n) such that $\bigcap V_i = p$, there is an integer N for which the image of $\pi_1(V_N - M)$ in $\pi_1(V_1 - M)$, obtained by inclusion, is cyclic or trivial.*

The next lemma occurs in [2], the proof being included here for completeness.

LEMMA 2. *If P is a finite polyhedron in the n -sphere S^n , then $\pi_1(S^n - P)$ is finitely generated.*

PROOF. Let N be a regular neighborhood of P in S^n ; then $S^n - \text{Int } N$ is a deformation retract of $S^n - P$ and thus $\pi_1(S^n - P)$ is isomorphic to $\pi_1(S^n - \text{Int } N)$. But $S^n - \text{Int } N$ is a finite polyhedron and hence has finitely generated fundamental group.

LEMMA 3. *If a k -manifold M in E^n , $k \leq n-2$, is locally tame at $p \in M$, then for any $\epsilon > 0$, there is a closed neighborhood V of p (in E^n) of diameter less than ϵ such that $\pi_1(V - M)$ is finitely generated.*

PROOF. Since M is locally tame at p , there is a neighborhood U of p and a homeomorphism $g: U \rightarrow E^n$ such that $g(U \cap M)$ is a polyhedron. If we could show that $g(p)$ had arbitrarily small neighborhoods V' for which $\pi_1(V' - g(U \cap M))$ is finitely generated, $g^{-1}(V')$ would then be the desired neighborhood of p . Thus we need only prove the lemma for M a polyhedron, an assumption we now make.

Let K be a complex with subcomplex L such that $|K| = E^n$, $|L| = M$, and p is a vertex of K . Since the closure of $M - |St(p, L)|$ is a closed set missing p there is a δ such that the δ -neighborhood $S(p, \delta)$ of p (in E^n) misses $\text{Cl}(M - |St(p, L)|)$; i.e., $S(p, \delta) \cap M = S(p, \delta) \cap |St(p, L)|$. Now let K' be a subdivision of K of mesh less than $\min\{\delta/2, \epsilon/2\}$ and define V to be $|St(p, K')|$; then V is a neighborhood of p and $V \cap M = V \cap |St(p, L)|$. From this and the definition of V it is not difficult to see that $V \cap M (= V \cap |St(p, L)|)$ is the join of $(\text{bdry } V) \cap M (= (\text{bdry } V) \cap |St(p, L)|)$ with p . We may thus define a deformation retraction of $V - M$ onto $(\text{bdry } V) - M$ by projection from p ; hence $\pi_1(V - M)$ is isomorphic to $\pi_1((\text{bdry } V) - M)$.

But $\text{bdry } V = \text{bdry} |\text{St}(p, K')|$ is a combinatorial $(n-1)$ -sphere and $(\text{bdry } V) \cap \bar{M}$ is a finite subpolyhedron so by Lemma 2, $\pi_1((\text{bdry } V) - M)$ is finitely generated. Finally, since V is the star of a vertex in a complex of mesh less than $\epsilon/2$, the diameter of V is less than ϵ .

A slight modification of the proof to Theorem 1 of [1] establishes the following lemma. Let a and b be the points in E^n defined by $a_i = b_i = 0$ for $i < n$, and $a_n = 1$, $b_n = -1$.

LEMMA 4. *If C is a compact set in E^{n-1} and T is the join of C with $a \cup b$, then the injection $i_*: \pi_1(E^{n-1} - C) \rightarrow \pi_1(E^n - T)$ is an isomorphism onto (i_* induced by inclusion).*

We now describe, for future reference, a sequence of closed neighborhoods of the vertex a , where a , b , c , and T are as in Lemma 4 and it is assumed that C contains the origin. The set A_i is to be the n -cube with "bottom" face parallel to E^{n-1} , having the vertex a at its center and edge length $2\gamma_i$, where $\gamma_i = \text{diam}(T \cap P_i)$, P_i being the horizontal $(n-1)$ -plane defined by $x_n = i/i+1$. The following facts follow from the construction:

- (i) $A_1 \supset A_2 \supset \dots$,
- (ii) $\bigcap A_i = p$,
- (iii) $\pi_1(A_1 - T)$ is isomorphic to $\Pi_1(E^n - C)$,
- (iv) the injection $i_*: \pi_1(A_j - T) \rightarrow \pi_1(A_1 - T)$ is an isomorphism onto.

Moreover, it is clear that the A_i 's would still satisfy (i)–(iv) if T were replaced with the cone over C with vertex a .

3. Proof of the Theorem.

(3.1) The starting point of all of our examples is the Fox-Artin mildly wild arc (Example 1.4 of [4]) with its end-points joined by polygonal arcs to a point q to form a 1-sphere. S is the union of the two tame arcs D_1 and D_2 running from q to the bad point p of the original arc. We note the following facts:

- 1. $\pi_1(E^3 - S)$ is infinitely generated.
- 2. The taming homeomorphism for each D_i may be chosen so that it is the identity outside some 3-cell.
- 3. S fails to be locally tame at p .

Fox and Artin actually show (3) in the process of proving that their arc is wild and an examination of the proof will yield a presentation of $\pi_1(E^3 - S)$ with generators c_i , $i = 1, 2, \dots$, and relations $c_i c_{i+1} c_i = c_{i+1} c_i c_{i+1}$; then, the map carrying c_i to the cycle $(i \ i+1)$ defines a homomorphism of $\pi_1(E^3 - S)$ onto an infinitely generated subgroup of the permutations of the positive integers. Thus S satisfies (1) as

claimed. Claim (2) follows from a result of Moise [6, Theorem 5].

(3.2) We establish Part I of the Theorem by proving the following

PROPOSITION 1. *For every $n \geq 3$, there is an $(n-2)$ -sphere S in E^n such that:*

- (i) $\pi_1(E^n - S)$ is infinitely generated.
- (ii) $S = D_1 \cup D_2$, where each D_i is a flat $(n-2)$ -cell with taming homeomorphism ϕ_i which is the identity outside some n -cell F_i .
- (iii) $D_1 \cap D_2 = \text{bdry } D_1 = \text{bdry } D_2$ is a flat polyhedral $(n-3)$ -sphere.

PROOF (BY INDUCTION ON n). The 1-sphere of (3.1) satisfies (i)–(iii), so the proposition is true for $n = 3$.

Suppose now that S is an $(n-3)$ -sphere in E^{n-1} satisfying conditions (i)–(iii). Let $T = \text{Susp}(S)$, i.e., the join of S with $a \cup b$ as in Lemma 4; T is an $(n-2)$ -sphere and by Lemma 4, $\pi_1(E^n - T)$ is isomorphic to $\pi_1(E^{n-1} - S)$ which is infinitely generated by hypothesis. Thus T satisfies (i). Moreover, $T = \text{Susp}(D_1) \cup \text{Susp}(D_2)$ and $\text{Susp}(D_i)$ is an $(n-2)$ -cell. Now the taming homeomorphism $\phi_i: F_i \rightarrow F_i$ is the identity on the boundary of F_i so the extension $\bar{\phi}_i: \text{Susp}(F_i) \rightarrow \text{Susp}(F_i)$ will be the identity on the boundary of the n -cell $\text{Susp } F_i$. We may thus extend $\bar{\phi}_i$ to all of E^n by letting it be the identity off $\text{Susp}(F_i)$ and $\bar{\phi}_i$ carries $\text{Susp}(D_i)$ onto the suspension of an $(n-3)$ -simplex which is of course a flat $(n-2)$ -cell; thus T satisfies (ii). Finally $\text{Susp}(D_1) \cap \text{Susp}(D_2) = \text{Susp}(D_1 \cap D_2)$ is a flat polyhedral $(n-4)$ -sphere. Thus T satisfies (iii) and the induction is complete.

Since condition (i) contradicts the conclusion of Lemma 2, S cannot be tame and we have proved Part I of the Theorem.

(3.3) Let S be an $(n-3)$ -sphere satisfying (i)–(iii) as in Proposition 1 and let D be $C(S)$, the cone over S with vertex a . $D = C(D_1) \cup C(D_2)$ and $C(D_i)$, by the arguments given for (ii) above, will be a flat $(n-2)$ -cell; similarly $C(D_1) \cap C(D_2)$ will be a flat $(n-3)$ -cell.

To see that D is wild we show that it fails to be locally tame at the vertex a . Let A_i be the closed neighborhoods of a as defined after Lemma 4, and let ϵ be small enough so that any neighborhood of diameter less than ϵ will be contained in A_1 . Then any such neighborhood V will contain some A_N for N large enough (recall that $A_1 \supset A_2 \supset \cdots$ and $\bigcap A_i = a$). The commutativity of the inclusion diagram

$$\begin{array}{ccc} A_N & \xrightarrow{\quad} & V \\ i \swarrow & & \searrow j \\ & A_1 & \end{array}$$

implies the commutativity of the induced injection diagram

$$\begin{array}{ccc}
 \pi_1(A_N - D) & \xrightarrow{\quad} & \pi_1(V - D) \\
 i_* \searrow & & \nearrow j_* \\
 & \pi_1(A_1 - D) &
 \end{array}$$

and since i_* is onto, j_* must be onto. But $\pi_1(A_1 - D) \approx \pi_1(E^{n-1} - S)$ is infinitely generated. We have shown then that for every neighborhood V of p of diameter less than ϵ , $\pi_1(V - D)$ is infinitely generated; thus, by Lemma 3, D cannot be locally tame at a and hence cannot be tame. This established Part II of the Theorem.

(3.4) PROPOSITION 2. *For every $n \geq 3$, there is an $(n-2)$ -sphere S in E^n such that:*

- (i) $\pi_1(E^n - S)$ is infinitely generated.
- (ii) S is locally flat except at a single point p .
- (iii) S fails to be locally tame at p .

PROOF (INDUCTION ON n). The 1-sphere given in (3.1) satisfies the conditions of the proposition which is thus true for $n=3$.

Suppose now that S' is an $(n-3)$ -sphere in E^{n-1} satisfying conditions (i)–(iii) above. Let $T = \text{Susp}(S')$ and $L = \text{Susp}(p')$. By Lemma 4, $\pi_1(E^n - T)$ is isomorphic to $\pi_1(E^{n-1} - S')$ and hence is infinitely generated. Clearly T will be locally flat except possibly at the points of the arc L . Since L is cellular in both E^n and T , shrinking L to a point will result in an $(n-2)$ -sphere $S = T/L$ in a space E^n/L which is homeomorphic to E^n ; let f be the shrinking map of E^n onto E^n/L . T is locally flat off L so S will be locally flat except possibly at the point $p = f(L)$, recalling that f is a homeomorphism off L . And hence $\pi_1(E^n - S)$ will be infinitely generated because f will induce an isomorphism carrying it onto $\pi_1(E^n - T)$; we have shown then that S satisfies (i).

Now consider the following sequence of closed neighborhoods of the arc L : $V_i = A_i \cup B_i \cup F_i$, where A_i is the neighborhood of the vertex a as described after Lemma 4, B_i is the closed $(1/i+1)$ -neighborhood of the vertex b , F_i is the suspension of an $(n-1)$ -cell containing all of S' in its interior, and F_i is the suspension of the closed $(1/i)$ -neighborhood of the bad point p' of S' (in E^{n-1}), $i \neq 1$. Clearly each V_i is a closed neighborhood of L , $\cap V_i = L$, and $V_1 \supset V_2 \supset \dots$. Moreover the commutativity of the diagram

$$\begin{array}{ccc}
 A_m & \xrightarrow{\quad} & V_m \\
 i \searrow & & \nearrow j \\
 & V_1 &
 \end{array}$$

implies the commutativity of the injection diagram

$$\begin{array}{ccc}
 \pi_1(A_m - T) & \xrightarrow{\quad} & \pi_1(V_m - T) \\
 i_* \searrow & & \nearrow j_* \\
 & \pi_1(V_1 - T) &
 \end{array}$$

and since $\pi_1(V_1 - T) \approx \pi_1(E^n - T)$ and i_* is onto, j_* maps $\pi_1(V_m - T)$ onto the infinitely generated group $\pi_1(V_1 - T)$. Defining $V'_i = f(V_i)$ we obtain a sequence $V'_1 \supset V'_2 \supset \cdots$ of closed neighborhoods of p' such that $\bigcap V'_i = p'$; finally, the injection $i_*: \pi_1(V'_m - S) \rightarrow \pi_1(V'_1 - S)$ is onto and $\pi_1(V'_1 - S)$ is infinitely generated (remembering that f is a homeomorphism off L). Since this violates the conclusion of Lemma 1, S cannot be locally flat at p . Using this sequence of neighborhoods, the arguments used in (3.3) show that S fails to be locally tame at p and the induction is complete. This establishes Part III of the Theorem.

(3.5) To obtain an $(n-2)$ -cell as described in Part IV of the Theorem, we consider the sphere T constructed in (3.4) and remove a nice cell that misses L before we shrink L to a point. All the arguments above are still valid and the result will be an $(n-2)$ -cell D which is locally flat at every point except p , p is an interior point of D , and D fails to be locally tame at p . We have thus completed the proof of the Theorem.

CONJECTURE. *The cells of Part IV are mildly wild; i.e., there is an interior point P of the $(n-2)$ -cell such that any $(n-2)$ -cell subset having P on its boundary is tame.*

The author has since proved the conjecture for the case $n=4$, solving a problem proposed by R. H. Fox [5, Problem 21].

Added in proof. We have recently learned that Sossinskiĭ has independently obtained similar examples.

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