SOME RELATIONS ASSOCIATED WITH AN EXTENSION OF KOSHLIAKOV'S FORMULA

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It is known that the following three relations are equivalent:

(1) Functional equation for $\zeta^2(s)$

$$\pi^{-s} \left[\Gamma \left(\frac{s}{2} \right) \right]^2 \zeta^2(s) = \pi^{s-1} \left[\Gamma \left(\frac{1-s}{2} \right) \right]^2 [\zeta(1-s)]^2;$$

(2) Koshliakov's formula

$$\begin{aligned} \gamma - \log \left(\frac{4\pi}{\tau} \right) + 4 \sum_{n=1}^{\infty} d(n) K_0(2\pi\tau n) \\ = \tau^{-1} [\gamma - \log(4\pi\tau)] + 4\tau^{-1} \sum_{n=1}^{\infty} d(n) K_0 \left(\frac{2\pi}{\tau} n \right); \end{aligned}$$

(3) Voronoï's sum formula

$$\begin{aligned} -\frac{1}{4} f(0) + \sum_{n=1}^{\infty} d(n) f(n) &= \int_0^{\infty} (2\gamma + \log x) f(x) dx \\ &+ 4 \sum_{n=1}^{\infty} d(n) \int_0^{\infty} f(x) \left[K_0(4\pi(nx)^{1/2}) - \frac{\pi}{2} Y_0(4\pi(nx)^{1/2}) \right] dx. \end{aligned}$$

In fact, these were obtained in [5] by specializing parameters in relations which were proved equivalent. In the present note, we give a proof of the statement [5, p. 63] that two other special cases of (3), namely

$$\begin{aligned} (4) \quad \frac{\rho}{2\pi^2} \sum_{n=1}^{\infty} d(n) \frac{\log \left(\frac{n}{\rho} \right)}{n^2 - \rho^2} &= \frac{\gamma}{4} + \frac{1}{8} \log \rho + \frac{1}{8\pi^2 \rho} \log(4\pi^2 \rho) \\ &+ \sum_{n=1}^{\infty} d(n) K_0(4\pi(n\rho)^{1/2}) \end{aligned}$$

and

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$$(5) \quad \rho \sum_{n=1}^{\infty} \frac{d(n)}{\rho^2 + n^2} = \frac{1}{4\rho} + \gamma\pi + \frac{1}{2} \pi \log \rho$$

$$+ 2\pi \sum_{n=1}^{\infty} d(n) \{ K_0(4\pi(-in\rho)^{1/2}) + K_0(4\pi(in\rho)^{1/2}) \}$$

can also be considered equivalent to (3). Thus any one of the relations (1) through (5) implies the others. However, as in [5], we shall treat the problem in a somewhat generalized form.

In the course of our investigation, we shall need the following results:

$$(6) \quad \int_0^{\infty} x^{\rho-1} K_0(xy) dx = \frac{1}{4} \left(\frac{2}{y} \right)^{\rho} \left[\Gamma \left(\frac{\rho}{2} \right) \right]^2, \quad [1, \text{p. 127}],$$

$$(7) \quad \int_0^{\infty} K_0 \left(\frac{a}{x} \right) K_0(xy) dx = \frac{\pi}{y} K_0(2(ay)^{1/2}), \quad [1, \text{p. 146}],$$

$$(8) \quad \int_0^{\infty} x K_0(ax) K_0(xy) dx = \frac{\log \left(\frac{a}{y} \right)}{a^2 - y^2},$$

$$(9) \quad \int_0^{\infty} \log x x^{\rho-1} K_0(xy) dx$$

$$= \frac{1}{4} \left(\frac{2}{y} \right)^{\rho} \left[\Gamma \left(\frac{\rho}{2} \right) \right]^2 \left\{ \log \left(\frac{2}{y} \right) + \psi \left(\frac{\rho}{2} \right) \right\}.$$

(8) can easily be obtained from [1, p. 145] and (9) from (6) above.

$$(9a) \quad \int_0^{\infty} K_0(ax) \left[\frac{2}{\pi} K_0((xy)^{1/2}) - Y_0((xy)^{1/2}) \right] dx = \frac{1}{a} K_0 \left(\frac{y}{4a} \right)$$

[5, p. 51].

1. We assume that

$$(a) \quad \begin{aligned} a_1, a_2 \dots; & \quad 0 < \lambda_1 < \lambda_2 \dots \rightarrow \infty, \\ b_1, b_2 \dots; & \quad 0 < l_1 < l_2 \dots \rightarrow \infty \end{aligned}$$

are four sequences of numbers such that the series

$$\sum_{n=1}^{\infty} \frac{|a_n|}{\lambda_n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{|b_n|}{l_n^2}$$

are convergent.

(b) $\alpha, \alpha', \beta, \beta'$ are four numbers such that the following relation holds

$$(I) \quad \alpha[\gamma + \log(\pi\tau)] - \alpha' + \sum_{n=1}^{\infty} a_n K_0(2\pi\lambda_n\tau) \\ = \frac{1}{\tau} \left\{ \beta \left[\gamma + \log \left(\frac{\pi}{\tau} \right) \right] - \beta' + \sum_{n=1}^{\infty} b_n K_0 \left(\frac{2\pi l_n}{\tau} \right) \right\}.$$

If we multiply both sides of (I) by $\tau K_0(2\pi\tau y)$, ($y > 0$), and integrate with regard to τ from zero to infinity, we get

$$(10) \quad [\alpha\gamma + \alpha \log \pi - \alpha'] \frac{1}{4\pi^2 y^2} + \alpha \frac{1}{4\pi^2 y^2} [-\log \pi y - \gamma] \\ + \frac{1}{4\pi^2} \sum_{n=1}^{\infty} a_n \frac{\log \left(\frac{\lambda_n}{y} \right)}{\lambda_n^2 - y^2} \\ = [\beta\gamma + \beta \log \pi - \beta'] \frac{1}{4y} + \frac{\beta}{4y} [\gamma + \log (4\pi y)] \\ + \frac{1}{2y} \sum_{n=1}^{\infty} b_n K_0(4\pi(l_n y)^{1/2}).$$

The interchange of the order of summation and integration is justified because of absolute convergence.

We rewrite (10) as

$$(II) \quad \frac{1}{2\pi^2 y} [-\alpha \log y - \alpha'] + \frac{y}{2\pi^2} \sum_{n=1}^{\infty} a_n \frac{\log \frac{\lambda_n}{y}}{\lambda_n^2 - y^2} \\ = \left[\beta\gamma + \frac{1}{2} \beta \log (4\pi^2 y) - \frac{1}{2} \beta' \right] + \sum_{n=1}^{\infty} b_n K_0(4\pi(l_n y)^{1/2}).$$

By analytic continuation (II) holds for all y such that $-\pi < \arg y < \pi$. Likewise, if we multiply (I) by $(1/\tau^2) K_0(2\pi y/\tau)$ and integrate, we get

$$(II') \quad \left(\alpha\gamma + \frac{1}{2} \alpha \log (4\pi^2 y) - \frac{1}{2} \alpha' \right) + \sum_{n=1}^{\infty} a_n K_0(4\pi(\lambda_n y)^{1/2}) \\ = \frac{1}{2\pi^2 y} (-\beta \log y - \beta') + \frac{y}{2\pi^2} \sum_{n=1}^{\infty} b_n \frac{\log \left(\frac{l_n}{y} \right)}{l_n^2 - y^2}.$$

If we replace y by $e^{i\tau}y$ in (II), we get the relation

$$\begin{aligned}
 & \frac{1}{2\pi^2 y} [\alpha \log (ye^{i\pi}) + \alpha'] + \frac{y}{2\pi^2} \sum_{n=1}^{\infty} a_n \frac{\log \left(\frac{\lambda_n}{ye^{i\pi}} \right)}{y^2 - \lambda_n^2} \\
 (11) \quad & = \left[\beta\gamma + \frac{1}{2} \beta \log (4\pi^2 ye^{i\pi}) - \frac{1}{2} \beta' \right] + \sum_{n=1}^{\infty} b_n K_0(4\pi(l_n ye^{i\pi})^{1/2}), \\
 & \qquad \qquad \qquad -2\pi < \arg y < 0.
 \end{aligned}$$

Adding the corresponding sides of (II) and (11), we obtain

$$\begin{aligned}
 & \frac{i\alpha}{2\pi y} - \frac{iy}{2\pi} \sum_{n=1}^{\infty} \frac{a_n}{y^2 - \lambda_n^2} \\
 (III) \quad & = 2\beta\gamma + \beta \log (4\pi^2 y) - \beta' + \frac{1}{2} i\pi\beta \\
 & \quad + \sum_{n=1}^{\infty} b_n \{ K_0(4\pi(l_n y)^{1/2}) + K_0(4\pi(l_n ye^{i\pi})^{1/2}) \}, \quad -\pi < \arg y < 0.
 \end{aligned}$$

The corresponding relation derived from (II') is

$$\begin{aligned}
 & 2\alpha\gamma + \alpha \log (4\pi^2 y) - \alpha' + \frac{1}{2} i\pi\alpha \\
 (III') \quad & \quad + \sum_{n=1}^{\infty} a_n \{ K_0(4\pi(\lambda_n y)^{1/2}) + K_0(4\pi(\lambda_n ye^{i\pi})^{1/2}) \} \\
 & = \frac{i\beta}{2\pi y} - \frac{iy}{2\pi} \sum_{n=1}^{\infty} \frac{b_n}{y^2 - \lambda_n^2}, \quad -\pi < \arg y < 0.
 \end{aligned}$$

2. Let

$$\sigma(y) = \frac{1}{2\pi i} \left\{ \sum_{n=1}^{\infty} a_n \frac{2y}{y^2 - \lambda_n^2} - 2 \frac{\alpha}{y} \right\}.$$

Then from (III)

$$\begin{aligned}
 \frac{1}{2} \sigma(y) & = 2\beta\gamma + \beta \log (4\pi^2 y) - \beta' + \frac{1}{2} i\pi\beta \\
 & \quad + \sum_{n=1}^{\infty} b_n \{ K_0(4\pi(l_n y)^{1/2}) + K_0(4\pi(l_n ye^{i\pi})^{1/2}) \}, \quad -\pi < \arg y < 0, \\
 & = - \left[2\beta\gamma + \beta \log (4\pi^2 y) - \beta' \right] + \frac{1}{2} i\pi\beta \\
 & \quad - \sum_{n=1}^{\infty} b_n \{ K_0(4\pi(l_n y)^{1/2}) + K_0(4\pi(l_n ye^{-i\pi})^{1/2}) \}, \quad 0 < \arg y < \pi.
 \end{aligned}$$

Let $\phi(y)$ be a function of the complex variable $y = u + iv$, regular in a strip $u \geq 0$, $|v| \leq \delta$, for some $\delta > 0$ and satisfying the following conditions:

$$(12) \quad (i) \int_0^\infty |\phi(u+iv)| du \text{ and } \int_0^\infty \log(u+iv) \phi(u+iv) du \text{ converge in}$$

$$-\delta < v < \delta.$$

(ii) There exists a sequence of the numbers u_n such that

$$(13) \quad \lim_{n \rightarrow \infty} \sigma(u_n + iv) \phi(u_n + iv) = 0 \text{ uniformly in } -\delta < v < \delta.$$

Since $\sigma(y)$ is analytic except for simple poles at zero and $\pm \lambda_n$, $n = 1, 2, 3, \dots$, we obtain by Cauchy's theorem and (13)

$$\sum_{n=1}^{\infty} a_n \phi(\lambda_n) = \int_C \sigma(y) \phi(y) dy$$

where C is the contour shown. Now we let $\epsilon \rightarrow 0$.

$$0 < \epsilon < \min(\delta, \lambda_1).$$

Thus

$$\begin{aligned} & \sum_{n=1}^{\infty} a_n \phi(\lambda_n) \\ &= \alpha \phi(0) + 4 \int_0^\infty [2\beta\gamma - \beta' + \beta \log(4\pi^2 u)] \phi(u) du \\ &+ \lim_{\epsilon \rightarrow 0} \left\{ 2 \int_{-i\epsilon}^{\infty - i\epsilon} \phi(y) \sum_{n=1}^{\infty} b_n [K_0(4\pi(l_n y)^{1/2}) + K_0(4\pi(l_n y e^{i\pi})^{1/2})] dy \right. \\ &\quad \left. + 2 \int_{i\epsilon}^{\infty + i\epsilon} \phi(y) \sum_{n=1}^{\infty} b_n [K_0(4\pi(l_n y)^{1/2}) + K_0(4\pi(l_n y e^{-i\pi})^{1/2})] dy \right\}. \end{aligned}$$

Let $\phi(y)$ be such that we can interchange the order of summation and integration. Then proceeding to the limit as $\epsilon \rightarrow 0$; and using

$$K_0(y e^{i\pi/2}) + K_0(y e^{-i\pi/2}) = -\pi Y_0(y),$$

we obtain

$$\begin{aligned}
 (IV) \quad & -\alpha\phi(0) + \sum_{n=1}^{\infty} a_n\phi(\lambda_n) = 4 \int_0^{\infty} [2\beta\gamma - \beta' + \beta \log(4\pi^2 u)]\phi(u) du \\
 & + 4 \sum_{n=1}^{\infty} b_n \int_0^{\infty} [K_0(4\pi(l_n u)^{1/2}) - \frac{\pi}{2} Y_0(4\pi(l_n u)^{1/2})]\phi(u) du.
 \end{aligned}$$

Likewise from (III'), we get the Sum-formula

$$\begin{aligned}
 (IV') \quad & -\beta\phi(0) + \sum_{n=1}^{\infty} b_n\phi(l_n) = 4 \int_0^{\infty} [2\alpha\gamma - \alpha' + \alpha \log(4\pi^2 u)]\phi(u) du \\
 & + 4 \sum_{n=1}^{\infty} a_n \int_0^{\infty} [K_0(4\pi(\lambda_n u)^{1/2}) - \frac{\pi}{2} Y_0(4\pi(\lambda_n u)^{1/2})]\phi(u) du.
 \end{aligned}$$

If in (IV) or (IV'), we let

$$\begin{aligned}
 \phi(u) &= K_0(2\pi u\tau) - K_0(2\pi u), \quad 0 < u < \infty, \\
 \phi(0) &= \log \frac{1}{\tau},
 \end{aligned}$$

and further assume that

$$\begin{aligned}
 (c) \quad & \alpha[\gamma + \log \pi] - \alpha' + \sum_{n=1}^{\infty} a_n K_0(2\pi\lambda_n) \\
 & = \beta[\gamma + \log \pi] - \beta' + \sum_{n=1}^{\infty} b_n K_0(2\pi l_n).
 \end{aligned}$$

Then using (6), (9), (9a), we get the relation (I).

Thus we have proved the following:

1. Under condition (a), each one of the six relations (II), (II'), (III), (III'), (IV), (IV') is a consequence of (I).
2. Under conditions (a) and (c), the relation (I) through (IV') are equivalent to each other.

If we set

$$\begin{aligned}
 a_n &= b_n = d(n); & \lambda_n &= l_n = n, \\
 \alpha &= \beta = \frac{1}{4}; & \alpha' &= \beta' = \frac{1}{2} \log(2\pi),
 \end{aligned}$$

in relations (I)–(IV') and note that conditions (a) and (c) are satisfied, we get the corresponding special cases mentioned in the introduction.

3. Now we study the functions $f(s)$ and $g(s)$ defined by:

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}, \quad \operatorname{Re} s \geq 2,$$

$$g(s) = \sum_{n=1}^{\infty} \frac{b_n}{l_n^s}, \quad \operatorname{Re} s \geq 2,$$

when conditions (a) and (b) hold.

Let

$$\mu(y) = \sum_{n=1}^{\infty} b_n \{ K_0(4\pi(l_n y)^{1/2}) + K_0(4\pi(l_n y e^{i\pi})^{1/2}) \}, \quad -\pi < \arg y < 0,$$

then

$$\mu(e^{-i\pi}y) = \sum_{n=1}^{\infty} b_n \{ K_0(4\pi(l_n y e^{-i\pi})^{1/2}) + K_0(4\pi(l_n y)^{1/2}) \}, \quad 0 < \arg y < \pi.$$

We denote the contour opposite as C_ν .

For $0 < \nu < \lambda_1$ and $\phi(y) = y^{-s}$, $\operatorname{Re} s \geq 2$, we obtain

$$\begin{aligned} f(s) &= \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s} = \int_{C_\nu} \sigma(y) y^{-s} dy \\ &= 4 \int_{\nu}^{\infty} [2\beta\gamma - \beta' + \beta \log(4\pi^2 u)] u^{-s} du + 2 \int_{\nu}^{\nu-i\epsilon} \frac{\mu(y)}{y^s} dy \\ &\quad + 2 \int_{\nu}^{\nu+i\epsilon} \frac{\mu(e^{-i\pi}y)}{y^s} dy + 2 \int_{\nu-i\epsilon}^{\infty-i\epsilon} \frac{\mu(y)}{y^s} dy + 2 \int_{\nu+i\epsilon}^{\infty+i\epsilon} \frac{\mu(e^{i\pi}y)}{y^s} dy. \end{aligned}$$

Making use of the fact that

$$\sum_{n=1}^{\infty} b_n K_0(4\pi(l_n z)^{1/2}) = O(\exp[-4\pi(l_1 \rho)^{1/2} \cos \theta/2])$$

where $z = \rho e^{i\theta}$, we replace the path of integration in the last two integrals by ones at right-angles to them. Thus

$$\begin{aligned}
 (14) \quad f(s) = & 4[2\beta\gamma - \beta' + \beta \log(4\pi^2)] \frac{\nu^{1-s}}{s-1} + 4\beta \log \nu \frac{\nu^{1-s}}{s-1} \\
 & + 4\beta \frac{\nu^{1-s}}{(s-1)^2} + 2 \int_{\nu}^{\nu-i\infty} \frac{\mu(y)}{y^s} dy + 2 \int_{\nu}^{\nu+i\infty} \frac{\mu(e^{-i\pi}y)}{y} dy.
 \end{aligned}$$

But the above integrals define a function of s which is regular for all finite values of s . Therefore $f(s)$ is analytic and single valued in the whole s -plane except perhaps at $s=1$, where it may have a pole of the second order, with the principal part

$$\frac{8\beta\gamma - 4\beta' + 4\beta \log(4\pi^2)}{s-1} + \frac{4\beta}{(s-1)^2}.$$

Likewise, using (III'), which is a consequence of (b), we can show that $g(s)$ has the same properties as $f(s)$, and that the principal part of $g(s)$ at $s=1$ is

$$\frac{4\alpha}{(s-1)^2} + \frac{8\alpha\gamma - 4\alpha' + 4\alpha \log(4\pi^2)}{s-1}.$$

Now if we proceed to the limit as $\nu \rightarrow 0$ in $\text{Re } s < 1$, we obtain from (14)

$$\begin{aligned}
 (15) \quad f(s) &= 2 \int_0^{-i\infty} \frac{\mu(y)}{y^s} dy + 2 \int_0^{i\infty} \frac{\mu(e^{-i\pi}y)}{y^s} dy \\
 &= 4 \sin\left(\frac{\pi}{2}s\right) \int_0^{\infty} \frac{\mu(e^{-i\pi/2}t)}{t^s} dt.
 \end{aligned}$$

This gives the integral representation of $f(s)$ in the half s -plane.

If we interchange the order of integration and summation in (15) and use

$$\int_0^{\infty} K_0(at^{1/2})t^{-s} dt = a^{2s-2}2^{1-2s}[\Gamma(1-s)]^2, \quad \text{Re } s < 1; \quad |\arg a| < \frac{\pi}{2},$$

we obtain

$$f(s) = \left[\sin\left(\frac{\pi}{2}s\right) \right]^2 [\Gamma(1-s)]^2 2^{2s-2} g(1-s) \pi^{2s-2}, \quad \text{Re } s < -1,$$

which can be written as

$$(16) \quad \pi^{-s} \left[\Gamma\left(\frac{s}{2}\right) \right]^2 f(s) = \pi^{-1+s} \left[\Gamma\left(\frac{1-s}{2}\right) \right]^2 g(1-s)$$

and by analytic continuation equality (16) holds for all s .

Finally, expanding both sides of (16) around $s = 0$ and $s = 1$ respectively, we find that

$$(17) \quad \alpha = f(0), \quad \alpha' = f'(0), \quad \beta = g(0), \quad \beta' = g'(0).$$

Thus we have proved the following: Under conditions (a) and (b), $(s-1)^2 f(s)$ and $(s-1)^2 g(s)$ are entire functions; $f(s)$ and $g(s)$ satisfy the functional equation (16) and further $\alpha, \alpha', \beta, \beta'$ are related to these functions as given in (17).

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