# A COMPARISON THEOREM FOR ELLIPTIC DIFFERENTIAL EQUATIONS

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Clark and the author [2] recently obtained a generalization of the Hartman-Wintner comparison theorem [4] for a pair of self-adjoint second order linear elliptic differential equations. The purpose of this note is to extend this generalization to general second order linear elliptic equations. As in [2], the usual pointwise inequalities for the coefficients are replaced by a more general integral inequality. The result is new even in the one-dimensional case, and extends Leighton's result for self-adjoint ordinary equations [5].

Protter [6] obtained pointwise inequalities in the nonself-adjoint case in two dimensions by the method of Hartman and Wintner [4]. We obtain an alternative to Protter's result as a corollary of our main theorem.

Let $R$ be a bounded domain in $n$-dimensional Euclidean space with boundary $B$ having a piecewise continuous unit normal. The linear elliptic differential operator $L$ defined by

$$Lu = \sum_{i,j=1}^{n} D_i(a_{ij}D_ju) + 2\sum_{i=1}^{n} b_iD_iu + cu, \qquad a_{ij} = a_{ji} \tag{1}$$

will be considered in $R$, where $D_i$ denotes partial differentiation with respect to $x^i$, $i=1, 2, \cdots, n$. We assume that the coefficients $a_{ij}$, $b_i$, and $c$ are real and continuous on $\overline{R}$, the $b_i$ are differentiable in $R$, and that the symmetric matrix $(a_{ij})$ is positive definite in $R$. A "solution" $u$ of $Lu=0$ is supposed to be continuous on $\overline{R}$ and have uniformly continuous first partial derivatives in $R$, and all partial derivatives involved in (1) are supposed to exist, be continuous, and satisfy $Lu=0$ in $R$.

Let $Q[z]$ be the quadratic form in $(n+1)$ variables $z_1, z_2, \cdots, z_{n+1}$ defined by

$$Q[z] = \sum_{i,j=1}^{n} a_{ij}z_iz_j - 2z_{n+1}\sum_{i=1}^{n} b_iz_i + gz_{n+1}^2, \tag{2}$$

where the continuous function $g$ is to be determined so that this form is positive semidefinite. The matrix $Q$ associated with $Q[z]$ has the block form

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$$Q = \begin{pmatrix} A & -b \\ -b^T & g \end{pmatrix}, \qquad A = (a_{ij}),$$

where $b^T$ is the $n$-vector $(b_1, b_2, \cdots, b_n)$. Let $B_i$ denote the cofactor of $-b_i$ in $Q$. Since $A$ is positive definite, a necessary and sufficient condition for $Q$ to be positive semidefinite is det $Q \geqq 0$, or

$$g \det (a_{ij}) \geqq - \sum_{i=1}^{n} b_i B_i. \tag{3}$$

The proof is a slight modification of the well-known proof for positive definite matrices [3, p. 306].

Let $J$ be the quadratic functional defined by

$$J[u] = \int_R F[u]\, dx \tag{4}$$

where

$$F[u] = \sum_{i,j} a_{ij} D_i u D_j u - 2u \sum_i b_i D_i u + (g - c)u^2,$$

with domain $\mathfrak{D}$ consisting of all real-valued continuous functions on $\overline{R}$ which have uniformly continuous first partial derivatives in $R$ and vanish on $B$.

Lemma. *Suppose $g$ satisfies* (3). *If there exists $u \in \mathfrak{D}$ not identically zero such that $J[u] < 0$, then every solution $v$ of $Lv = 0$ vanishes at some point of $\overline{R}$.*

Proof. Suppose to the contrary that there exists a solution $v \neq 0$ in $\overline{R}$. For $u \in \mathfrak{D}$ define

$$X^i = v D_i(u/v);$$

$$Y^i = v^{-1} \sum_j a_{ij} D_j v, \qquad i = 1, 2, \cdots, n;$$

$$E[u, v] = \sum_{i,j} a_{ij} X^i X^j - 2u \sum_i b_i X^i + gu^2 + \sum_i D_i(u^2 Y^i).$$

A routine calculation yields the identity

$$E[u, v] = F[u] + u^2 v^{-1} Lv.$$

Since $Lv = 0$ in $R$,

$$
J[u] = \int_R \Big[ \sum_{i,j} a_{ij} X^i X^j - 2u \sum_i b_i X^i + g u^2 \Big] dx
\tag{5}
$$
$$
+ \int_R \sum_i D_i(u^2 Y^i)\, dx.
$$

Since $u=0$ on $B$, the second integral is zero by Green's formula. The first integrand is a positive semidefinite form by hypothesis (3). The contradiction $J[u] \geqq 0$ establishes the lemma.

Consider in addition to (1) a second differential operator $L^*$ of the same form,

$$
\overset{*}{L} u = \sum_{i,j=1}^{n} D_i(\overset{*}{a}_{ij} D_j u) + 2 \sum_i \overset{*}{b}_i D_i u + \overset{*}{c} u, \qquad \overset{*}{a}_{ij} = \overset{*}{a}_{ji}
$$

in which the coefficients satisfy the same conditions as the coefficients in (1). $L^*$ is the Euler-Jacobi operator associated with the quadratic functional $J^*$ defined by

$$
\overset{*}{J}[u] = \int_R \Big[ \sum_{i,j} \overset{*}{a}_{ij} D_i u D_j u - 2u \sum_i \overset{*}{b}_i D_i u - c^* u^2 \Big] dx.
\tag{6}
$$

Define $V[u] = J^*[u] - J[u]$, $u \in \mathfrak{D}$. Since $u=0$ on $B$, it follows from partial integration that

$$
V[u] = \int_R [\sum (\overset{*}{a}_{ij} - a_{ij}) D_i u D_j u
\tag{7}
$$
$$
+ \{ \sum D_i(b_i^* - b_i) + c - c^* - g \} u^2 ]\, dx.
$$

THEOREM 1. *Suppose* $g$ *satisfies* (3). *If there exists a nontrivial solution* $u$ *of* $L^*u=0$ *in* $R$ *such that* $u=0$ *on* $B$ *and* $V[u]>0$, *then every solution of* $Lv=0$ *vanishes at some point of* $\bar{R}$.

PROOF. The hypothesis $V[u]>0$ is equivalent to $J[u]<J^*[u]$. Since $u=0$ on $B$, it follows from Green's formula that $J^*[u]=0$. Hence the hypothesis $J[u]<0$ of the lemma is fulfilled.

THEOREM 2. *Suppose* $g \det (a_{ij}) > -\sum b_i B_i$. *If there exists a nontrivial solution of* $L^*u=0$ *in* $R$ *such that* $u=0$ *on* $B$ *and* $V[u] \geqq 0$, *then every solution of* $Lv=0$ *vanishes at some point of* $\bar{R}$.

Since $Q$ is positive definite, the lemma is valid when the hypothesis $J[u]<0$ is replaced by $J[u] \leqq 0$. The proof of Theorem 2 is then analogous to that of Theorem 1.

In the case that equality holds in (3), that is

$$g = -\sum_i b_i B_i/\det(a_{ij}), \tag{8}$$

define

$$\delta = \sum D_i(b_i^* - b_i) + c - c^* - g.$$

$L$ is called a "strict Sturmian majorant" of $L^*$ by Hartman and Wintner [4] when the following conditions hold: (i) $(a_{ij}^* - a_{ij})$ is positive semidefinite and $\delta \geqq 0$ in $\bar{R}$; (ii) either $\delta > 0$ at some point or $(a_{ij}^* - a_{ij})$ is positive definite and $c^* \neq 0$ at some point. The corollary below follows immediately from Theorem 1.

COROLLARY. *Suppose that $L$ is a strict Sturmian majorant of $L^*$. If there exists a solution $u$ of $L^*u = 0$ in $R$ such that $u = 0$ on $B$ and $u$ does not vanish in any open set contained in $R$, then every solution of $Lv = 0$ vanishes at some point of $\bar{R}$.*

If the coefficients $a_{ij}^*$ are of class $C^{2,1}(R)$ (i.e. all second derivatives exist and are Lipschitzian), the hypothesis that $u$ does not vanish in any open set of $R$ can be replaced by the condition that $u$ does not vanish identically in $R$ because of Aronszajn's unique continuation theorem [1].

In the case $n = 2$ considered by Protter [6], the condition $\delta \geqq 0$ reduces to

$$(a_{11}a_{22} - a_{12}^2)\left(\sum_{i=1}^{2} D_i(b_i^* - b_i) + c - c^*\right) \geqq a_{11}b_2^2 - 2a_{12}b_1b_2 + a_{22}b_1^2, \tag{9}$$

which is considerably simpler than Protter's condition. It reduces to Protter's condition

$$\sum_{i=1}^{2} D_i b_i^* + c - c^* \geqq 0$$

in the case that $b_1 = b_2 = 0$, and also in the case that $a_{12} = a_{12}^* = 0$, $a_{11} = a_{11}^*$, $a_{22} = a_{22}^*$. (Two incorrect signs appear in [6]).

The following example in the case $n = 2$ illustrates that Theorem 1 is more general than the pointwise condition (9). Let $R$ be the square $0 < x^1, x^2 < \pi$. Let $L^*$, $L$ be the elliptic operators defined by

$$L^*u = D_1^2 u + D_2^2 u + 2u,$$

$$Lv = D_1^2 v + D_2^2 v + D_1 v + cv,$$

where

$$c(x^1, x^2) = f(x^1)f(x^2) + 5/4,$$

and $f \in C[0, \pi]$ is not identically zero. The function $u = \sin x^1 \sin x^2$ is zero on $B$ and satisfies $L^*u = 0$. The condition $V[u] > 0$ of Theorem 1 reduces to

$$\int_0^\pi \int_0^\pi f(x^1)f(x^2) \sin^2 x^1 \sin^2 x^2 \, dx^1 dx^2 > 0.$$

Since this is fulfilled, every solution of $Lv = 0$ vanishes at some point of $\bar{R}$. This cannot be concluded from (9) or from Protter's result [**6**] unless $f$ has constant sign in $R$.

In the case $n = 1$, $L$ is an ordinary differential operator of the form

$$Lu = (au')' + 2bu' + cu,$$

and $R$ is an interval $(x_1, x_2)$. We assert that $\bar{R}$ can be replaced by $R$ in the lemma and theorems; for $v$ can have at most a simple zero at the boundary points $x_1$ and $x_2$, and hence the first integral on the right side of (5) exists and is nonnegative provided only that $v \neq 0$ in $R$.

THEOREM 3. *If there exists a nontrivial solution $u$ of $L^*u = 0$ in $R$ such that $u = 0$ on $B$ and*

$$\int_{x_1}^{x_2} \left[ (a^* - a)u'^2 + \left( b^{*\prime} - b' + c - c^* - \frac{b^2}{a} \right) u^2 \right] dx > 0, \tag{10}$$

*then every solution of $Lv = 0$ has a zero in $(x_1, x_2)$.*

In the self-adjoint case $b = b^* = 0$ it was shown by Clark and the author [**2**] that the strict inequality in the hypothesis $V[u] > 0$ of Theorem 1, and therefore also in (10), can be replaced by $\geqq$. Indeed, this is transparent when the proof of the above lemma is specialized to the self-adjoint case. With $>$ replaced by $\geqq$, (10) reduces to Leighton's condition in the self-adjoint case [**5**].

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# $q$-ANALOGUES OF CAUCHY'S FORMULAS

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1. Let $q$ be a given number and let $\alpha$ be real or complex. The $\alpha$th "basic number" is defined by means of $[\alpha]=(1-q^\alpha)/(1-q)$. This has served as a basis for an extensive amount of literature in mathematics under such titles as Heine, basic, or $q$-series and functions. The basic numbers also occur naturally in many theta identities. The works of Jackson (for bibliography see [2]) and Hahn [3] have stimulated much interest in this field.

One important operation that is intimately connected with basic series as well as with difference and other functional equations is the $q$-derivative of a function $f$. This is defined by

$$Df(x) = \frac{f(qx) - f(x)}{x(q-1)}. \tag{1.1}$$

Jackson defined the operations, which he called $q$-integration,

$$\int_0^x f(t)d(q,t) = x(1-q)\sum_{k=0}^{\infty} q^k f(xq^k) \tag{1.2}$$

and

$$\int_x^\infty f(t)d(q,t) = x(1-q)\sum_{k=0}^{\infty} q^{-k} f(xq^{-k}) \tag{1.3}$$

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