A DYNAMIC PROGRAMMING GENERALIZATION OF xy TO n VARIABLES

A. P. HILLMAN, D. G. MEAD, K. B. O'KEEFE¹ AND E. S. O'KEEFE

A function $\phi(d_1, \dots, d_n)$, from the ordered n -tuples of nonnegative integers into the nonnegative integers augmented by ∞ , with the following properties arises in the investigation [6] of the differential ideal generated by a product of n differential indeterminates:

- (A) $\phi(d_1, \dots, d_n)$ is symmetric in the d_i .
- (B) $\phi(0) = 0$ and $\phi(x) = \infty$ for $x > 0$.
- (C) $\phi(x, y) = xy$.
- (D) $\phi(d_1, \dots, d_n)$

$$= \min [\phi(d_1 - r, \dots, d_a - r) + \phi(d_{a+1} - s, \dots, d_b - s) + \dots \\ + \phi(d_{c+1} - t, \dots, d_n - t) + \phi(r, s, \dots, t)]$$

where $r, s, \dots, t$ range over all nonnegative integers such that the arguments are nonnegative.

Many other properties of ϕ can be derived from these four, including the following:

(E) $\phi(d_1, \dots, d_n) = \min [\phi(d_1 - t, \dots, d_{n-1} - t) + td_n].$

(F) $\phi(d_1, \dots, d_n) = \min [(d_{n-1} - T_{n-2})(d_n - T_{n-2}) + \sum_{i=1}^{n-2} t_i(d_i - T_{i-1})],$

where $T_i = t_1 + \dots + t_i$.

Property (F) shows that ϕ is a solution of a quadratic integer programming problem. Some references are given at the close of this paper; others may be found in these listed publications.

We assume using (A) that the d_i are numbered so that $d_1 \leq d_2 \leq \dots \leq d_n$ and then use (C) and (E) to find an explicit expression for ϕ .

Let $D = (d_1, \dots, d_n)$. For $2 \leq i \leq n$ let $q_i = q_i(D) = (d_1 + \dots + d_i) \cdot (i-1)^{-1}$ and let $k = k(D)$ be the smallest i for which q_i assumes its minimum value. Let integers $q = q(D)$ and $r = r(D)$ be defined by $d_1 + \dots + d_k = q(k-1) + r$ and $0 \leq r < k-1$. Let $c_i = q - d_i$ for $i = 1, \dots, k$ and let $C = C(D) = (c_1, \dots, c_k; r)$. Let $s_1 = c_1 + \dots + c_k$ and $s_2 = \sum_{i < j} c_i c_j$. We define a function $f(D)$ by

Received by the editors April 3, 1965.

¹ Work on this paper supported by a grant from the American Association of University Women.

$$f(D) = s_2 + rs_1 + [(r+1)r/2]$$

and show by induction on n that $f(D) = \phi(D)$, i.e., $f(D)$ satisfies (E) and (C).

If $d_1, \dots, d_n$ are allowed to take on all real values, it can be shown easily by induction on n that the function $\phi(D)$ satisfying (C) and (E), with t ranging over the real numbers, is

$$\phi(D) = \sum_{i < j} d_i d_j - \frac{n-2}{2(n-1)} (d_1 + \dots + d_n)^2.$$

Going back to the integer case, we note that $s_1 = (q-d_1) + \dots + (q-d_k) = kq - [(k-1)q+r] = q-r$ and hence that

$$(1) \quad q(D) = s_1(D) + r(D).$$

We also observe that q_i is the average of the $i-1$ integers $d_1 + d_2, d_3, \dots, d_i$. Therefore one and only one of the following holds for each $i \geq 3$:

$$(2) \quad (a) \ q_{i-1} < q_i < d_i, \quad (b) \ q_{i-1} = q_i = d_i, \quad (c) \ q_{i-1} > q_i > d_i.$$

Since the d 's are nondecreasing, if $q_j \leq q_{j+1}$ for some j then (2) implies that $q_j \leq q_{j+1} \leq q_{j+2} \leq \dots \leq q_n$. The first such j must therefore be k . Hence we have

$$(3) \quad q_2 > q_3 > \dots > q_k \quad \text{and} \quad q_k \leq q_{k+1} \leq \dots \leq q_n.$$

It follows from (2) and (3) that

$$(4) \quad d_i < q_i \leq q_{i+1} \leq d_{i+1} \quad \text{if and only if } i = k.$$

Since $(k-1)q+r = d_1 + \dots + d_k = (k-1)q_k$ and $0 \leq r < k-1$, we also note that q is the greatest integer $[q_k]$ in q_k .

In the case $n=2$, we always have $k=2$, $q=d_1+d_2$, $r=0$, $C=(d_2, d_1; 0)$ and $f(d_1, d_2) = s_2 = d_1 d_2$. Hence $f(d_1, d_2) = \phi(d_1, d_2)$. We assume that $f(D) = \phi(D)$ for $n=m$ and show that it follows for $n=m+1$.

When $n=m+1$, we let $d_{m+1}=d$, $D^*=(d_1, \dots, d_m, d)$, and $D=(d_1, \dots, d_m)$. Using the hypothesis of the induction and (E) we see that $\phi(D^*)$ is

$$(5) \quad \min[f(D-tI) + td]$$

where t ranges over $1, 2, \dots, d_1$ and $D-tI$ denotes $(d_1-t, \dots, d_m-t)$.

Let $F(t) = f(D-tI) + td$ and let $\nabla F(t) = F(t) - F(t-1)$. Clearly $\nabla F(t) = d - \Delta f(D-tI)$ where $\Delta f(D) = f(D+I) - f(D)$. We first prove the following:

LEMMA. $q(D+I) - q(D)$ is 1 if $r(D+I) > 0$ and is 2 if $r(D+I) = 0$, $\Delta f(D) = q(D) + 1$, and $\nabla F(t) = d - q(D - tI) - 1$.

PROOF. Let $C(D) = (c_1, \dots, c_k; r)$. When $k(D+I) = k(D)$, it is clear that if $r < k - 2$ then $q(D+I) = q(D) + 1$ and that $C(D+I) = (c_1, \dots, c_k; r+1)$ while $q(D+I) = q(D) + 2$ and $C(D+I) = (c_1 + 1, \dots, c_k + 1; 0)$ if $r = k - 2$. In each of these cases, one calculates that $f(D+I) = f(D) + s_1(D) + r(D) + 1$. This and (1) lead to $\Delta f(D) = q(D) + 1$. Then $\nabla F(t) = d - \Delta f(D - tI) = d - q(D - tI) - 1$.

Next we note that

$$(6) \quad q_i(D+I) = q_i(D) + 1 + 1/(i-1).$$

Since $1/(i-1)$ is a decreasing function of i for $i \geq 2$, this implies that $k(D+I) \geq k(D)$. Hence the only case remaining is that in which $k(D+I) > k = k(D)$. We then have

$$(7) \quad q_{k+1}(D) \geq q_k(D), \quad q_{k+1}(D+I) < q_k(D+I).$$

Using (6) and (7) we obtain

$$(8) \quad 0 < q_k(D+I) - q_{k+1}(D+I) \\ = \frac{1}{k-1} - \frac{1}{k} - q_{k+1}(D) - q_k(D) \leq \frac{1}{(k-1)k}.$$

Since q_{k+1} and q_k are both expressible as integers divided by $(k-1)k$, (8) implies that $q_k(D+I) - q_{k+1}(D+I) = 1/(k-1)k$ and

$$(9) \quad q_{k+1}(D) = q_k(D).$$

One sees from (9) and (2) that $d_{k+1} = q_k(D)$. One similarly finds that $d_i = q_k(D)$ for $k < i \leq k(D+I)$.

Since $q_k(D) = d_{k+1}$ is an integer, we have $q(D) = q_k(D)$ and $r(D) = 0$. Hence $d_i = q(D)$ for $k+1 \leq i \leq k(D+I)$ and it is easily seen that $q(D+I) = q(D) + 1$, $r(D+I) = r(D) + 1 = 1$, and $C(D+I) = (c_1, \dots, c_k, 0, \dots, 0; 1)$. One now calculates that $f(D+I) - f(D) = s_1(D) + 1 = q(D) + 1$. As before, $\nabla F(t) = d - q(D - tI) - 1$, which completes the proof of the lemma.

It follows from the lemma that $\nabla F(t)$ increases by 1 or 2 and $q(D - tI)$ decreases by 1 or 2 when t increases by 1.

We prove that $f(D^*) = \phi(D^*)$ by considering cases. First let $d \geq q(D)$. For $1 \leq t \leq d_1$, we then have $d \geq q(D) > q(D - tI)$ and so $\nabla F(t) = d - q(D - tI) - 1 \geq 0$. Hence the minimum $\phi(D^*)$ of the $F(t)$ is $F(0) = f(D)$. If $k(D^*) > k(D)$, we must have $k(D) = m$ and $d \geq q(D)$ implies that $C(D)$ and $C(D^*)$ are of the form $(c_1, \dots, c_m; r)$ and

$(c_1, \dots, c_m, 0; r)$ respectively. If $k(D^*) = k(D)$, $C(D^*) = C(D)$. Either way, $f(D^*) = f(D) = \phi(D^*)$.

It remains to consider $d < q(D)$. Then $d_m \leq d < q(D) \leq q_k(D)$ and (4) implies that $k(D) = m$ and $k(D^*) = m + 1$. Now $d_1 > 0$ since $d_1 = 0$ would imply $k(D^*) = 2$, which is less than $m + 1$ for $m \geq 2$. Then $d \geq d_2 > d_2 - d_1 = q(D - d_1 I)$. Hence $d - q(D - tI)$ must change sign as t goes from 0 to d_1 in one of the two following ways:

First let there be a c with $0 < c < d_1$ such that $d = q(D - cI)$. Then $\nabla F(t) \leq -1$ for $t \leq c$ and $\nabla F(t) \geq 0$ for $t > c$. Hence $\phi(D^*) = F(c) = f(D - cI) + cd$. We let $k = k(D - cI)$ and show that $k = m$. Since $c > 0$,

$$q_k(D - cI) \leq d_{k+1} - c < d = q(D - cI) \leq q_k(D - cI)$$

would be a contradiction unless $k = m$ (and d_{k+1} does not exist). Now it is clear that $q(D^*) = q(D - cI) + c$. Letting $C(D - cI) = (c_1, \dots, c_m; r)$, one has $C(D^*) = (c_1, \dots, c_m, c; r)$ and $f(D^*) = f(D - cI) + c(c_1 + \dots + c_m) + cr = f(D - cI) + cd = \phi(D^*)$.

Finally let there be a c with $0 \leq c < d_1$, $d = q[D - (c+1)I] + 1$, and $d = q(D - cI) - 1$. Then $\nabla F(t) \leq 0$ for $t < c$ and $\nabla F(t) \geq 2$ for $t \geq c$. Hence $\phi(D^*) = F(c) = f(D - cI) + cd$. Since $q(D - cI) - q[D - (c+1)I] = 2$, the lemma tells us that $r(D - cI) = 0$. As before, $k(D - cI) = m$. Let $C(D - cI) = (c_1, \dots, c_m; 0)$. One then calculates that $q(D^*) = q(D - cI) + c - 1$, $C(D^*) = (c_1 - 1, \dots, c_m - 1, c; m - 1)$, and $f(D^*) = f(D - cI) + (c_1 + \dots + c_m - 1)c = f(D - cI) + dc = \phi(D^*)$. This completes the proof.

BIBLIOGRAPHY

1. E. M. L. Beale, *On quadratic programming*, Naval Res. Logist. Quart. **6** (1959), 227-243.
2. R. Bellman, *Dynamic programming*, Princeton Univ. Press, Princeton, N. J., 1957.
3. ———, *Mathematical optimization techniques*, Univ. of California Press, Berkeley, Calif., 1965.
4. R. E. Gomory, *An algorithm for integer solutions to linear programs*, Recent advances in mathematical programming, pp. 269-302, McGraw-Hill, New York, 1963.
5. ———, *Mathematical programming*, Slaughter Memorial Paper No. 10, pp. 99-110, Math. Assoc. Amer., Buffalo, N. Y., 1965.
6. A. P. Hillman, D. G. Mead, K. B. O'Keefe and E. S. O'Keefe, *Ideals generated by products*, Proc. Amer. Math. Soc. **17** (1966), 715-718.
7. H. P. Kunzi and W. Oettle, *Integer quadratic programming*, Recent advances in mathematical programming, pp. 303-308, McGraw-Hill, New York.
8. P. Wolfe, *The simplex method for quadratic programming*, Econometrica, **27** (1959), 382-398.

UNIVERSITY OF SANTA CLARA AND
UNIVERSITY OF WASHINGTON, SEATTLE