IDEMPOTENTS IN GROUP RINGS

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The following result generalizes a corollary of a theorem due to Swan [**1**, p. 571].[1] Swan's theorem deals with Dedekind rings of characteristic 0.

THEOREM. *Let R be an integral domain* (*with* 1), *and let G be a finite group of order n. Then the group ring RG has an idempotent different from* 0 *or* 1 *if and only if some prime divisor of n is a unit in R.*

PROOF. Suppose that p is a prime dividing n, and that p is a unit in R. Let P be a subgroup of G of order p, and let $e=p^{-1}\sum_{x\in P} x$. Then e is a nontrivial idempotent.

Conversely, suppose that $e=\sum_{g\in G}\alpha_g g$ is a nontrivial idempotent in RG and that the characteristic of R is zero. Let $E=(e_{ij})$ be the matrix representing e under the regular representation. Then $e_{ii}=\alpha_1$ for $i=1,\cdots,n$, so the trace of E is $n\alpha_1=q$, the multiplicity of 1 as an eigenvalue of E. Since $e\neq 0$, 1, we have that $0<q<n$. Let n_1 and q_1 be the relatively prime integers such that n_1 divides n, q_1 divides q, and $n_1\alpha_1=q_1$. Let a and b be integers such that $an_1+bq_1=1$; then $(a+b\alpha_1)n_1=1$, so that n_1 is a unit in R. Since $q<n$, we see that $n_1\neq 1$. Any prime divisor of n_1 is a unit in R.

Next, let R have characteristic $p\neq 0$, and let $n=p^r t$, where $(p,t)=1$ and $r\geqq 0$.

It is well known that in the group algebra of a p-group over a field of characteristic p, every zero divisor is nilpotent. In particular, this is the case in KG if $t=1$, where K denotes the quotient field of R.

Suppose that RG has a nontrivial idempotent e. Then e is a non-nilpotent zero divisor in RG (hence in KG). Thus by the above, $t\neq 1$. Clearly t is a unit in R.

BIBLIOGRAPHY

1. R. G. Swan, *Induced representations and projective modules*, Ann. of Math. **71** (1960), 552–578.

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[1] The author is indebted to the referee for this reference.