ON SPLITTING FIELDS FOR CERTAIN LIE ALGEBRAS OF PRIME CHARACTERISTIC

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1. Let F be a field of prime characteristic different from 2 or 3 and L a Lie algebra over F with an abelian Cartan subalgebra H . For α in H^* (the dual space of H) set $L_\alpha = \{x \in L \mid [xh] = \alpha(h)x, \text{ for all } h \text{ in } H\}$, and as usual, if $L_\alpha \neq (0)$, α is called a root with respect to H and L_α the root space for α . We have $L_0 = H$ and $[L_\alpha L_\beta] \subset L_{\alpha+\beta}$. Seligman and Mills in [1] have called L a Lie algebra of classical type if L contains an abelian Cartan subalgebra H and if H and L satisfy:

- (i) $[LL] = L$.
- (ii) L has center (0).
- (iii) L is a direct sum of subspaces L_α .
- (iv) If α is a nonzero root, then $[L_\alpha L_{-\alpha}]$ is one-dimensional.
- (v) If α is a nonzero root and $\beta \in H^*$, then there is a positive integer m such that $\beta + m\alpha$ is not a root.

Let L be a Lie algebra over F such that L_K is of classical type, where K is the algebraic closure of F . An extension field P of F is called a splitting field for L provided L_P is of classical type. We can now state the main theorem of this paper as:

THEOREM 1.1. *Every semisimple Lie algebra over F with nondegenerate Killing form $(x, y) = \text{tr}(\text{ad } x)(\text{ad } y)$ has a separable splitting field.*

Note that if F is finite and L has nondegenerate Killing form then every finite extension is separable, in particular one that splits L . Therefore, we may assume, in what follows, that F is infinite.

LEMMA 2.1. *If L is semisimple with nondegenerate Killing form, then L contains a regular element x such that the minimum polynomial of $\text{ad}(x)$ has the form:*

$$\mu(\lambda) = \lambda \prod_{\alpha} (\lambda - \alpha(x))$$

where the $\alpha(x)$ are distinct and different from zero in some extension P of F .

PROOF. Recall that an element x in L is regular provided the 0-space of $\text{ad}(x)$ has minimal dimension. If x is regular in L and P is

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an extension of F then x is regular in L_P . To see this, let $(u_1, u_2, \dots, u_n)$ be a basis for L and $(X_1, X_2, \dots, X_n)$ be algebraically independent indeterminants. Let $F' = F(X_1, \dots, X_n)$. Then $X = \sum X_i u_i$ is in $L_{F'}$ and the characteristic polynomial of $\text{ad}(X)$ is given by:

$$\det(\lambda I - \text{ad}(X)) = \sum_0^n M_i(X) \lambda^i,$$

where $M_i(X) \in F[X_1, \dots, X_n]$. If $x = \sum_1^n \xi_i u_i$, then $M_i(\xi) = 0$ for $i < r$, where r is the dimension of the zero space of $\text{ad}(x)$, and $M_r(\xi) \neq 0$. Also, since x is regular, $M_i(\eta_1, \eta_2, \dots, \eta_n) = 0$ for $i < r$ and all η_i in F . Thus, since F is infinite, $M_i(X)$ is zero for $0 \leq i < r$ and

$$\det(\lambda I - \text{ad}(X)) = \sum_r^n M_i(X) \lambda^i.$$

Now let $y = \sum \mu_j u_j$, μ_j in P , i.e. y in L_P . Then $M_i(\mu) = 0$ for $0 \leq i < r$, so that the zero space of $\text{ad}(y)$ in L_P has dimension greater than or equal to r . Therefore x is regular in L_P as claimed. As in the above, for generic element X in $L_{F'}$ we have:

$$\det(\lambda I - \text{ad}(X)) = \lambda^r (M_r(X) + M_{r+1}(X)\lambda + \dots + M_n(X)\lambda^{n-r}),$$

where $M_r(X) \neq 0$. Consider $g(X, \lambda)$, where:

$$g(X, \lambda) = M_r(X) + M_r(X)\lambda + \dots + M_n(X)\lambda^{n-r}.$$

Then $g(X, \lambda) \in F[X_1, \dots, X_n, \lambda] \subset F(X_1, \dots, X_n)[\lambda] = F'[\lambda]$, and thus g has a discriminant given by:

$$D(X_1, \dots, X_n) = \left(\prod_{i < j} (\rho_i - \rho_j) \right)^2,$$

where $\rho_1, \rho_2, \dots, \rho_{n-r}$ are all roots of $g(X, \lambda)$ as a polynomial in λ in some splitting field over F' , multiple ones taken as many times as their multiplicity. Now, $D(X_1, \dots, X_n)$ is a symmetric function of the roots and therefore is in the ring generated by the elementary functions of the roots, i.e. the ring generated by $M_r(X)$, $M_{r+1}(X)$, $\dots$, $M_{n-r}(X)$. Thus, there exists a polynomial $Q(y_r, y_{r+1}, \dots, y_n)$ with integral coefficients such that

$$Q(M_r(X), \dots, M_n(X)) = 0$$

if and only if $g(X, \lambda)$ has repeated roots in its splitting field. Consider $q(X) = M_r(X) Q(M_r(X), \dots, M_n(X))$. Then $q(X) \in F[X_1, \dots, X_n]$ and if $q(X) \neq 0$ there exist elements $\xi_1, \dots, \xi_n$ in F such that $q(\xi_1, \dots, \xi_n) \neq 0$. Suppose such an n -tuple exists and set $x = \sum \xi_i u_i$,

$x \in L$. Then $\text{ad}(x)$ has characteristic polynomial

$$\det(\lambda I - \text{ad}(x)) = \lambda^r (M_r(\xi) + M_{r+1}(\xi)\lambda + \cdots + \lambda^{n-r}).$$

$M_r(\xi) \neq 0$ and $M_r(\xi) + M_{r+1}(\xi)\lambda + \cdots + M_n(\xi)\lambda^{n-r}$ has distinct roots in a splitting field. Thus x is regular and the minimum polynomial of $\text{ad}(x)$ has the form:

$$\mu(\lambda) = \lambda \prod_{\alpha} (\lambda - \alpha(x)),$$

where $\alpha(x)$ are distinct and different from zero. It remains to show that $q(X) \neq 0$. For this, let H be a standard Cartan subalgebra in L_K and $h_0 \in H$ be such that $\alpha(h_0)$ are distinct and nonzero for all roots α relative to H . If $h_0 = \sum \omega_i u_i$, $\omega_i \in K$, then $q(\omega_1, \dots, \omega_n)$ is not zero, so that $q(X_1, \dots, X_n) \neq 0$, as desired.

3. Proof of Theorem 1.1. Let L be a semisimple Lie algebra over F of dimension n , with nondegenerate Killing form, and x a regular element, where the dimension of the zero-space of $\text{ad}(x)$ is r and where the minimum polynomial of $\text{ad}(x)$ has the form:

$$(1) \quad \mu_x(\lambda) = \lambda \prod_{\alpha} (\lambda - \alpha(x)), \quad \alpha(x) \in F,$$

with all $\alpha(x)$ distinct, different from zero, and $n-r$ in number. We will show that L is of classical type. Note that (ii) is satisfied by our hypotheses. Let H be the zero space of $\text{ad}(x)$. Then H is the Cartan subalgebra of L which will play the role of satisfying the remaining axioms, and H has dimension r . Since all $\alpha(x)$ are distinct and characteristic roots of $\text{ad}(x)$, the subspaces $L_{\alpha(x)}$ corresponding to $\alpha(x)$ have dimension one. Then we have

$$(2) \quad L = H + \sum_{\alpha} L_{\alpha}.$$

Now for $h \in H$, $[hx] = 0$ and if $y \in L_{\alpha}$, $[[yh]x] = [[yx]h] = \alpha(x)[yh]$. Thus $[yh] \in L_{\alpha}$, i.e. $[L_{\alpha}H] \subseteq L_{\alpha}$ for $\alpha(x) \neq 0$. Since L_{α} is one dimensional this means that for $e_{\alpha} \in L_{\alpha}$ and $h \in H$, $[e_{\alpha}h] = \lambda(h)e_{\alpha}$. Set $\alpha(h) = \lambda(h)$. Thus the characteristic roots of $\text{ad}(h)$ are in the ground field and L_{α} is a root space relative to H . Furthermore, the restriction to H of the Killing form on L is nondegenerate. To see this, let $h \in H$, $e_{\alpha} \in L_{\alpha}$, for $\alpha \neq 0$. Then $[e_{\alpha}h] = \alpha(h)e_{\alpha}$ and we can choose $e_{\alpha}^{(1)} \in L_{\alpha}$ such that $[e_{\alpha}^{(1)}h] = e_{\alpha}$. Then $(h, e_{\alpha}) = (h, [e_{\alpha}^{(1)}h]) = ([hh], e_{\alpha}^{(1)}) = 0$. Thus $(H, L_{\alpha}) = 0$ for all $\alpha \neq 0$ and the form must be nondegenerate on H . It follows that H is abelian. In the case where F is algebraically closed

this is a result due to Zassenhaus. In the general case a field extension argument gives the result. This, together with (2) now shows that axiom (iii) holds.

Thus we have that L contains an abelian Cartan subalgebra H , and that relative to a fixed basis for L , $\text{ad}(h)$ has a diagonal matrix for every $h \in H$. Next we note that if α is a root different from zero, then so is $-\alpha$. For, let $e_\alpha \in L_\alpha$, $e_\beta \in L_\beta$. Then, for some $h \in H$, $e_\alpha = [e_\alpha h]$, and $(e_\beta, e_\alpha) = (e_\beta, [e_\alpha h]) = ([e_\beta e_\alpha], h) = 0$, unless $\beta = -\alpha$, since $[e_\beta e_\alpha] \in L_{\alpha+\beta}$. If $L_{-\alpha} = (0)$ we have $(L, e_\alpha) = 0$, a contradiction.

The nondegeneracy of the form (x, y) on H implies that for each $\alpha \in H^*$ there exists an $h_\alpha \in H$ such that $(h_\alpha, h) = \alpha(h)$, for all $h \in H$.

In what follows we shall have occasion to refer to results in Seligman's Memoir [2] which we shall denote by M.

LEMMA 3.1 (M, COROLLARY 3.2). *If $e_\alpha \in L_\alpha$, $e_{-\alpha} \in L_{-\alpha}$, then $[e_{-\alpha} e_\alpha] = (e_{-\alpha}, e_\alpha)h$.*

Since L has nondegenerate form, every derivation of L is inner [3]. Thus, for $x \in L$, $\text{ad}(x)^p$ is a derivation in L and there exists a unique $y \in L$ such that $\text{ad}(x)^p = \text{ad}(y)$. Setting $x^p = y$, L becomes a restricted Lie algebra over F and the adjoint mapping is a restricted representation of L .

We turn now to a modification of two results due to Jacobson dealing with low-dimensional Lie algebras and their representations (M, Lemma 4.1 and 4.2). The modification involves replacing algebraic closure of the ground field with the fact that for the representation U we have $U(h)$ is diagonalizable for all $h \in H$.

LEMMA 3.2. *Let L be a two-dimensional Lie algebra over F with basis elements e, h , and $[eh] = e$. Let U be an irreducible representation of L such that $U(h)$ and $U(e)^p$ are diagonalizable. Then either $U(e)^p = 0$ or U is equivalent to the p -dimensional representation W :*

$$W(e) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad W(h) = \begin{pmatrix} \lambda & & & & 0 \\ & \lambda + 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \lambda + p - 1 \end{pmatrix}.$$

PROOF. Suppose $U(e)^p = U(e^p) \neq 0$. Then, since $U(e)^p$ and $U(h)^p - U(h)$ are diagonalizable by our assumptions, these matrices are scalar. For, if $V_1 = \{v \mid vU(e)^p = \lambda v\}$, and $V_2 = \{v \mid v(U(h)^p - U(h)) = \mu v\}$, then these are invariant subspaces of the representation space V and since one of them is not zero for some λ by diagonalizability one must be the whole space. Thus in each case, $U(e)^p = \sigma I$, $\sigma \in F$,

and $U(h)^p - U(h) = \rho I$, $\rho \in F$. Now, let λ be a characteristic root of $U(h)$ and $v \neq 0$ such that $vU(h) = \lambda v$. Then $vU(e)^p = \sigma v$, and the space spanned by $\{v, vU(e), \dots, vU(e)^{p-1}\}$ is an invariant subspace of dimension p , thus the whole space V . To see the invariance, we note:

$$(3) \quad vU(e)^k U(h) = (\lambda + k)vU(e)^k, \quad 0 \leq k \leq p-1.$$

Now, relative to this basis for V , the matrices of $U(h)$ and $U(e)$ have the form of the lemma.

LEMMA 3.3. *Let L be a three-dimensional Lie algebra over F with basis e, f, h and let $[ef] = h$, $[fh] = 0 = [eh]$. Let U be a nonzero irreducible representation of L such that $U(e)^p = 0$ and $U(f)^p = 0$ and $U(h)$ is diagonalizable. Then $\text{tr}(u(e)U(f)) = 0$.*

PROOF. Since $U(h)$ is diagonalizable and centralizes the representation we have $U(h)$ is scalar, i.e. $U(h) = \lambda I$. If $\lambda = 0$, then $U(e)U(f) = U(f)U(e)$. Since both $U(e)$ and $U(f)$ are nilpotent, so is $U(e)U(f)$, and thus $\text{tr}(U(e)U(f)) = 0$. Suppose now that $\lambda \neq 0$ and let $v \neq 0$ be an element of the representation space V such that $vU(f) = 0$. Such a v exists since $U(f)$ is nilpotent. Now, let K be the space spanned by $\{v, vU(e), \dots, vU(e)^{p-1}\}$. Then $KU(e) \subset K$ and $KU(h) \subset K$. Furthermore, we have:

$$(4) \quad vU(e)^k U(f) = vU(e)^{k-1} U(f)U(e) + vU(e)^{k-1} U(h).$$

Actually, by induction we have:

$$(5) \quad vU(e)^k U(f) = k\lambda(vU(e)^{k-1}), \quad k \geq 1.$$

Thus, $K = V$, the whole space, and thus $\{v, vU(e), \dots, vU(e)^{p-1}\}$ is a basis for V . The matrices relative to this basis are:

$$U(e) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad U(f) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \lambda & 0 & 0 & \cdots & 0 \\ 0 & 2\lambda & & \cdots & 0 \\ 0 & 0 & \cdots & (p-1)\lambda & 0 \end{pmatrix}$$

and thus $\text{tr}(U(e)U(f)) = 0$ as claimed.

Using these lemmas together with our diagonalizability condition we can now prove the following analogues of the required theorems in M .

THEOREM 3.1 (M, THEOREM 4.1). *If $\alpha \neq 0$ is a root then $e_\alpha^2 = 0$.*

PROOF. For $h \in H$, $[he_\alpha^2] = 0$, thus $e_\alpha^2 \in H$. Choose h such that $\alpha(h) = 1$. Then $\{e_\alpha, h\}$ forms a two-dimensional Lie algebra L_1 as in Lemma 3.2. For the representation $U(x)$ take $\text{ad}_L(x)$. Then, the restriction of U to L_1 can be written in the form:

$$\begin{pmatrix} U_1 & & & \\ & U_2 & & 0 \\ & * & \ddots & \\ & & & U_t \end{pmatrix}.$$

Furthermore, since $\text{ad}_L(h)$ are diagonal for $h \in H$, the same holds for $\text{ad}_L(h)$ restricted to M , M an irreducible L_1 submodule of L , and for the transformations induced by $\text{ad}_L(h)$ in L/M . Continuing this argument on L/M we see that $U_i(h)$ is a diagonal matrix relative to a suitable basis for each $h \in H$.

Now $(e_\alpha^p, h) = \text{tr}(U(e_\alpha)^p U(h))$ and either $U_i(e_\alpha)^p = 0$ or $U_i(e_\alpha)$ has the form of Lemma 3.2. In any case, $\text{tr}(U_i(e_\alpha)^p U_i(h))$ is zero so that $(e_\alpha^p, h) = 0$. This holds whenever $\alpha(h) \neq 0$. If $\alpha(h) = 0$, let $h_1 \in H$ be chosen such that $\alpha(h_1) \neq 0$. Then $\alpha(h+h_1) \neq 0$ and $(e_\alpha^p, h) = (e_\alpha^p, h+h_1) - (e_\alpha^p, h_1) = 0$. Thus, $(e_\alpha^p, H) = 0$, which gives $e_\alpha^p = 0$.

THEOREM 3.2 (M, THEOREM 4.2). *If $\alpha \neq 0$ is a root, then $\alpha(h_\alpha) \neq 0$.*

PROOF. Suppose $\alpha(h_\alpha) = 0$ and $e_\alpha \neq 0$, $e_\alpha \in L_\alpha$, $e_{-\alpha} \neq 0$, $e_{-\alpha} \in L_{-\alpha}$ such that $(e_\alpha, e_{-\alpha}) = 1$. By Lemma 3.1 $[e_{-\alpha}e_\alpha] = h_\alpha$. Let L_1 be the algebra spanned by $\{e_\alpha, e_{-\alpha}, h_\alpha\}$. For the irreducible constituent U_i of the restriction of $U = \text{ad}_L$ to L_1 we have $U_i(e_\alpha)^p = 0$ and $U_i(e_{-\alpha})^p = 0$. Thus by Lemma 3.3 we obtain $\text{tr}(U_i(e_{-\alpha})U_i(e_\alpha)) = 0$. Therefore, $\text{tr}(U(e_{-\alpha})U(e_\alpha)) = 0$, i.e. $(e_{-\alpha}, e_\alpha) = 0$, a contradiction.

Now, by Lemma 3.1 and Theorem 3.2 we have $[L_\alpha L_{-\alpha}]$ is one dimensional, giving us axiom (iv). Theorems 3.1 and 3.2 make it possible now to use the results of §5 of M. (For a complete proof of M, Lemma 5.1 see [1].) In particular, axiom (v) for our algebras is a consequence of Theorems 5.2 and 5.4 in M.

Finally, for axiom (i), we note that $[LL]_K = [L_K L_K] = L_K$, K the algebraic closure of F . Hence we have:

$$\dim_F[LL] = \dim_K[LL]_K = \dim_K L_K = \dim_F L.$$

Therefore, $[LL] = L$. Thus, our algebra L is of classical type and this together with Lemma 2.1 proves Theorem 1.1.

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