A PARACOMPACT SEMI-METRIC SPACE WHICH IS NOT AN M_3-SPACE[1]

R. W. HEATH

E. A. Michael has shown that a regular Hausdorff space S is paracompact if and only if any of the following is true: (1) every open cover of S has a σ-locally finite open refinement [8], (2) every open cover of S has an open σ-closure preserving refinement [9], or (3) every open cover of S has an open σ-cushioned refinement [10]. The Nagata-Smirnov Theorem [12] or [14] (see also Bing's Theorem 3 in [1]) states that a T_3-space with a σ-locally finite base is metrizable.

In [3] Jack Ceder defines an *M_1-space* to be a T_3-space with a σ-closure preserving base and an *M_3-space* to be a T_1-space with a σ-cushioned pair base (see Definition 1 below). By Michael's theorems cited above both M_1- and M_3-spaces are paracompact, and in view of the Nagata-Smirnov Theorem, one might suspect that they would even be metrizable. In [3], however, Ceder shows that such is not the case; in fact M_1- and M_3-spaces need not be first countable, and they need not be metrizable even if they are first countable. Ceder does show, though, that a first countable M_3- (and hence M_1-) space is a paracompact semi-metric space, and he raises the question: is this a characterization—*i.e.* is every paracompact semi-metric space an M_3-space (and perhaps even an M_1-space)? A negative answer is given to that question in this paper.

Also it was called to my attention by Professor Michael that a question raised by C. Borges [2] is answered (in the negative) by the same example below, which is a cosmic space (the regular continuous image of a separable metric space [11]) that is not an M_3-space.

DEFINITION 1. Let P be a collection of ordered pairs of subsets of the T_1-space S such that, for each $p=(p_1, p_2)\in P$, p_1 is open and $p_1\subset p_2$, and such that, for every $x\in S$ and every neighborhood U of x, there is a $p\in P$ for which $x\in p_1\subset p_2\subset U$. Then P is called a *pair base* for S. Moreover, P is called *cushioned* if, for every $Q\subset P$,

$$\mathrm{Cl}\bigcup\{p_1: p\in Q\}\subset\bigcup\{p_2: p\in Q\}$$

and P is *σ-cushioned* if it is the union of countably many cushioned collections.

An M_3-space is a T_1-space with a σ-cushioned pair base.

Presented to the Society December 29, 1965; received by the editors June 19, 1965.

[1] The research for this paper was partially supported by National Science Foundation grant NSF G-23790.

Note that Borges calls M_3-spaces stratifiable spaces in [2]. Also, first countable M_3-spaces are sometimes called Nagata spaces [3].

DEFINITION 2. A T_1-space S is a *semi-metric space* provided that there is a function d from $S\times S$ into the nonnegative reals such that

(1) for every $(x, y)\in S\times S$, $d(x, y)=d(y, x)$ and $d(x, y)=0$ if and only if $x=y$ and

(2) for every $x\in S$ and $M\subset S$, $\inf\{d(x, y): y\in M\}=0$ if and only if $x\in \mathrm{Cl}\, M$ (*i.e.*, "the topology is invariant with respect to d").

All other terms are defined as in [7] or [13].

Alternative characterization of semi-metric spaces (Theorem 3.2 of [4]: A T_1-space S is semi-metric if and only if there is a collection $\{g(n, x): x\in S, n=1, 2, \cdots\}$ of open sets such that (1) for each $x\in S$, $\{g(n, x): n=1, 2, \cdots\}$ is a local base for the topology at x and (2) if $y\in S$ and x is a point sequence such that, for each m, $y \in g(m, x_m)$, then x converges to y.

THEOREM 1. *There exists a regular Lindelöf (and hence paracompact) semi-metric space S which is a cosmic space (the continuous image of a separable metric space) but is not an M_3-space.*

PROOF. Such a space S is defined as follows. The points of S are those points z of the complex plane such that either (1) $\mathrm{Im}\, z=0$ and $\mathrm{Re}\, z$ is irrational or (2) $\mathrm{Im}\, z>0$ and both $\mathrm{Im}\, z$ and $\mathrm{Re}\, z$ are rational. For each point $z\in S$ of type 1 (*i.e.*, $\mathrm{Im}\, z=0$) and each natural number n, $B(n, z)=\{x\in S: \mathrm{Im}\, x<|\mathrm{Re}(x-z)|<1/n \text{ or } z=x\}$ (*i.e.* the "bow-tie region" of radius $1/n$ and vertex angle 45°) is a basis element for S; and for each $z\in S$ of type 2 and for each n, the open disc of radius $1/n$ and center z is a basis element. The space is clearly regular since, for every $z\in S$ with $\mathrm{Im}\, z=0$ and for each n, $B(n, z)$ has only the two points $z+1/n$ and $z-1/n$ of S on its boundary (all points of its boundary in the complex plane having at least one coordinate irrational). Also S is easily seen to be semi-metric by the above alternative characterization, and S is clearly Lindelöf. That S is a cosmic space follows easily by the same argument given for Example 12.1 in [11]. Finally, S is not an M_3-space. For suppose that there were a σ-cushioned pair base $P=\bigcup\{Q(i): 1=1, 2, \cdots\}$ for S. Then there would be a second category subset M of the x-axis, with $M\subset S$, and natural numbers m and k such that, for each $x\in M$, there is a $q\in Q(k)$ such that

$$(*)\qquad B(m, x) \subset q_1 \subset q_2 \subset B(1, x).$$

Pick a rational number r in the closure (with respect to the Euclidean topology) of M. Then r is in the closure of $\{x\in M: x>r\}$ or of

$\{x \in M: x < r\}$; assume the former. Let $R \subset Q(k)$ consist of all $q \in Q(k)$ which satisfy (*) for some $x \in M$ with $r < x < r + 1/m$. Then, for $y = (r + 1/m, 1/m)$, $y \in \mathrm{Cl} \bigcup \{p_1 : p \in R\}$ but $y \notin \bigcup \{p_2 : p \in R\}$. That contradicts the assumption that $Q(k)$ is cushioned. Thus S is not an M_3-space.

The questions remain (1) whether every M_3-space is an M_1-space (see [3] for the definition of an M_2-space, an "intermediate" space), (2) what topological condition is necessary in order for a paracompact semi-metric space to be an M_3-space, (3) whether every Lindelöf semi-metric space is a cosmic space, (4) whether every separable M_3-space is a cosmic space (see [2]) and (5) whether every regular countable space is an M_3-space. For some theorems relating semi-metric and M_3-spaces see [5], and for a necessary and sufficient condition that an M_3-space be metrizable (namely: that it have a point-countable base) see [6].

References

1. R. H. Bing, *Metrization of topological spaces*, Canad. J. of Math. **3** (1951), 175–186.

2. C. J. R. Borges, *On stratifiable spaces*, Pacific J. Math. **17** (1966), 1–16.

3. Jack Ceder, *Some generalizations of metric spaces*, Pacific J. Math. **11** (1961), 105–125.

4. R. W. Heath, *Arc-wise connectedness in semi-metric spaces*, Pacific J. Math. **12** (1962), 1301–1319.

5. ———, *On open mappings and certain spaces satisfying the first countability axiom*, Fund. Math. **57** (1965), 91–96.

6. ———, *On spaces with point-countable bases*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **13** (1965), 393–395.

7. J. L. Kelley, *General topology*, Van Nostrand, Princeton, N. J., 1955.

8. E. A. Michael, *A note on paracompact spaces*, Proc. Amer. Math. Soc. **4** (1953), 831–838.

9. ———, *Another note on paracompact spaces*, Proc. Amer. Math. Soc. **8** (1957), 822–828.

10. ———, *Yet another note on paracompact spaces*, Proc. Amer. Math. Soc. **10** (1959), 309–314.

11. ———, *$\aleph_0$-spaces*, J. Math. Mech. (to appear).

12. J. Nagata, *On a necessary and sufficient condition of metrizability*, J. Inst. Polytech. Osaka City Univ. **1** (1950), 93–100.

13. R. L. Moore, *Foundations of point set theory*, rev. ed., Amer. Math. Soc. Colloq. Publ. Vol. 13, Providence, R. I., 1962.

14. Yu. M. Smirnov, *A necessary and sufficient condition for metrizability of a topological space*, Dokl. Acad. Nauk SSSR **77** (1951), 197–200.

Arizona State University and
University of Georgia