THE FREE MODULAR LATTICE, $FM(2+2+2)$, IS INFINITE

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The following are known concerning the cardinality of the free modular lattice generated by a set of distinct chains.

1. $FM(1+1+1)$ contains 28 elements [1, p. 68].
2. $FM(2+1+1)$ contains 138 elements [2, p. 1]; [3, p. 629].
3. $FM(1+1+1+1)$ contains an infinite chain of distinct elements [4, p. 114]. The proof is due to G. Birkoff.
4. $FM(n_1+n_2)$, n_1, n_2 finite, is a finite distributive lattice [1, p. 72].

By a method similar to the one used to prove 3 above, the following theorem is proved.

THEOREM. *$FM(2+2+2)$ contains an infinite chain of distinct elements.*

PROOF. Let G be the free Abelian group (additive) generated by $x_1, x_2, \dots, x_n, \dots$. For $k=0, 1, 2, \dots$ let us define the following.

$$a_1 = [x_{3k+1} + x_{3k+2}],$$

the subgroup generated by the set of all $x_{3k+1} + x_{3k+2}$;

$$a_2 = [x_{3k}, x_{3k+1} + x_{3k+2}],$$

$$b_1 = [x_{3k+2} + x_{3k+3}], \quad b_2 = [x_{3k+1}, x_{3k+2} + x_{3k+3}],$$

$$c_1 = [x_{3k+3} + x_{3k+4}], \quad c_2 = [x_{3k+2}, x_{3k+3} + x_{3k+4}].$$

Thus we have six subgroups of G in which $a_1 < a_2$, $b_1 < b_2$, $c_1 < c_2$. We let L be the lattice of subgroups generated by these six subgroups. Since G is an Abelian group, L is modular [1, p. 65].

Let us designate certain elements of L as follows where $k=0, 1, 2, \dots$.

$$T_1 = a_2 \cap b_2, \quad \text{and} \quad \text{for } n \geq 1,$$

$$T_{6n+1} = T_1 \cap T_{6n} = [x_{6n+3k+1} + x_{6n+3k+2} + x_{6n+3k+3}].$$

For $n \geq 0$,

$$T_{6n+2} = a_1 \cup T_{6n+1} = [x_{3k+1} + x_{3k+2}, x_{6n+3k+3}].$$

$$T_{6n+3} = (a_2 \cap c_2) \cap T_{6n+2} = [x_{6n+3k+3} + x_{6n+3k+4} + x_{6n+3k+5}].$$

$$T_{6n+4} = c_1 \cup T_{6n+3} = [x_{3k+3} + x_{3k+4}, x_{6n+3k+5}].$$

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$$T_{6n+5} = (b_2 \cap c_2) \cap T_{6n+4} = [x_{6n+3k+5} + x_{6n+3k+6} + x_{6n+3k+7}].$$

$$T_{6n+6} = b_1 \cup T_{6n+5} = [x_{3k+2} + x_{3k+3}, x_{6n+3k+7}].$$

From this it follows that $T_1 < T_7 < T_{13} < \dots < T_{6n+1} < \dots$. Since L is a modular lattice generated by a partly ordered set of the form $2+2+2$, $FM(2+2+2)$ contains an infinite chain.

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