SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

ABELIAN FORMAL GROUPS

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If A is a commutative ring with 1, a *formal group over A* is a formal power series $F(x, y)$ in two variables with coefficients in A satisfying the following two conditions (cf. [2]):

(i) $F(x, y) \equiv x + y \pmod{\text{degree } 2}$,

(ii) $F(F(x, y), z) = F(x, F(y, z))$.

$F(x, y)$ is *abelian* if $F(x, y) = F(y, x)$.

Lazard [1] has shown that all formal groups over A are abelian when $P(A) = 0$, where $P(A)$ denotes the ideal of nilpotent elements in A . It is the purpose of this note to prove the following extension and converse.¹

THEOREM. *In order that all formal groups over A be abelian it is necessary and sufficient that $P(A)$ be torsion-free as an additive group*

PROOF. If $P(A)$ is not torsion-free there is a nonzero $a \in P(A)$ and a prime p such that $pa = 0$. If $b = a^n$ for appropriate n , we have $b \neq 0$, $b^2 = 0$, $pb = 0$. A simple calculation shows that

$$F(x, y) = x + y + bx^py$$

is a nonabelian formal group.

Conversely assume that $P(A)$ is torsion-free and let $F = x + y + \sum_{i,j=1}^{\infty} a_{ij}x^i y^j$ be a formal group. (It is an easy consequence of (i) and (ii) that there are no terms in x^2 , y^2 , x^3 , $\dots$.) Taking the $a_{ij} \pmod{P(A)}$ makes F a formal group over $A/P(A)$ and since $P(A/P(A)) = 0$, Lazard's theorem tells us that $a_{ij} \equiv a_{ji} \pmod{P(A)}$. Thus to prove that $a_{ij} = a_{ji}$ it is sufficient to find a nonzero integer q such that $qa_{ij} = qa_{ji}$, and this we get by comparing the coefficients of $x^m y^n z^m$ for appropriate m, n in the two sides of the associative law (ii).

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¹ The referee informs me that a new proof of Lazard's theorem will appear in the mimeographed notes of the 1965-66 course given by J.-P. Serre at the Collège de France.

The term xyz yields $2a_{12}=2a_{21}$; inductively we assume that $a_{ij}=a_{ji}$ whenever $i, j < k$ and we wish to prove that $a_{km}=a_{mk}$ for $0 < m < k$.

Let $n=k-m$. The coefficient of $x^m y^n z^m$ in $F(F(x, y), z)$ is the coefficient of $x^m y^n$ in

$$\sum_{i=1}^k a_{im} \left(x + y + \sum_{j=1}^m \sum_{r=1}^n a_{jr} x^j y^r \right)^i.$$

(The limits of summation are large enough to guarantee that we have all contributions to the term in $x^m y^n$.) Similarly the coefficient of $x^m y^n z^m$ in $F(x, F(y, z))$ is the coefficient of $y^n z^m$ in

$$\sum_i a_{mi} \left(y + z + \sum_{j,r} a_{jr} y^j z^r \right)^i,$$

or, what is the same thing, the coefficient of $x^m y^n$ in

$$\sum_{i=1}^k a_{mi} \left(x + y + \sum_{j=1}^m \sum_{r=1}^n a_{rj} x^j y^r \right)^i.$$

The inductive hypothesis allows us to cancel all but one term when we compare the expressions for these coefficients, leaving us with

$$\binom{k}{m} a_{km} = \binom{k}{m} a_{mk},$$

and this completes the proof.

BIBLIOGRAPHY

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2. J. Lubin, *One-parameter formal Lie groups over p-adic integer rings*, Ann. of Math. (2) **80** (1964), 464-484.

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