argument (see [1, Lemma 2.4]) that this last series is finite.

As a corollary to the theorem we note that

$$\sum_{n=1}^{\infty} (-1)^n n^{-1} P(|S_n| > n\epsilon)$$

is convergent for all $\epsilon > 0$. In [2] it is shown that this series is absolutely convergent for all $\epsilon > 0$ if and only if $EX_1 = 0$.

References

1. M. Rosenblatt, *On the oscillation of sums of random variables*, Trans. Amer. Math. Soc. **72** (1952), 165–178.

2. F. Spitzer, *A combinatorial lemma and its application to probability theory*, Trans. Amer. Math. Soc. **82** (1956), 323–339.

University of Missouri and
University of New Mexico

THE BOUNDARY OF THE RANGE OF A VECTOR MEASURE

VOLKER BAUMANN

Let $(X, \mathbf{S})$ be a measure space, μ_i $(i=1, \cdots, n)$ signed measures on $(X, \mathbf{S})$. Then $\mu=(\mu_1, \cdots, \mu_n)$ is a (n-dimensional) *vector measure* on $(X, \mathbf{S})$, μ is *finite* and *purely nonatomic* if every μ_i is finite and purely nonatomic, respectively. Consider the range of a finite n-dimensional vector measure as a subset of the n-dimensional Euclidean space E^n. A. Liapounoff [4] and P. R. Halmos [2] have shown:

(1) *The range of a finite vector measure is closed,*

(2) *the range of a finite and purely nonatomic vector measure is convex.*

For any (infinite) vector measure μ call $R=\{\mu(M): M\in\mathbf{S}$ and $\mu(M)$ finite$\}$ the *finite range* of μ. Then it is an immediate consequence of (2) that

(3) *the finite range of a purely nonatomic vector measure is convex.*

Two simple examples due to R. Borges [1] however show that there are purely nonatomic as well as purely atomic vector measures the finite range of which is not closed:

(a) $\mathbf{S}$ is the σ-ring of the one-dimensional Lebesgue sets, μ_1 the Lebesgue measure and $\mu_2(M)=\int_M \exp(-z^2)dz$, $M\in\mathbf{S}$. The positive

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x_1-axis does not belong to the finite range R of (μ_1, μ_2), but it belongs to the closure $\overline{R}$ of R.

(b) $\boldsymbol{S}$ is the σ-ring of all subsets of the set of positive integers, μ the one-dimensional vector measure defined by $\mu(M) = \sum_{n\in M}(2-2^{-n})$, $M\in\boldsymbol{S}$. The set of the even positive integers does not belong to R, but it belongs to $\overline{R}$.

As to boundary points of the finite range R of a vector measure lying moreover on the boundary of the convex hull of R here it will be proved the

THEOREM. *Let π be a supporting hyperplane of the finite range R of a vector measure, $\overline{R}$ the closure of R. If $\pi\cap\overline{R}$ is bounded, then $\pi\cap\overline{R}\subset R$.*

Restricted to purely nonatomic vector measures the theorem was proved by R. Borges [1].

PROOF. Given π, there is a real linear function L on E^n and a real number u such that $\pi=\{\xi\in E^n: L(\xi)=u\}$ and

$$u = \inf\{\nu(M): M \in \boldsymbol{S}_0\}, \tag{4}$$

where $\nu(M)=L(\mu(M))$ and $\boldsymbol{S}_0=\{M\in\boldsymbol{S}: \mu(M) \text{ finite}\}$. Let $\xi\in\pi\cap\overline{R}$. Then there is a sequence $\{M_j\}$, $M_j\in\boldsymbol{S}_0$, $\lim \mu(M_j)=\xi$. We can assume that

$$\nu(M_j) \leqq u + 2^{-j} \qquad (j = 1, 2, \cdots). \tag{5}$$

Write $F_j^k=\bigcup_{i=j}^{j+k} M_i$. Since $\boldsymbol{S}_0$ is a ring (see [3, p. 19 and 119]) and ν is additive on $\boldsymbol{S}_0$ it is easy to verify by induction for k, using (4) and (5), that

$$u \leqq \nu(F_j^k) \leqq u + 2^{-j}\sum_{m=0}^{k} 2^{-m} \qquad (j = 1, 2, \cdots; k = 0, 1, \cdots).$$

Therefore for all j, k

$$u \leqq \nu(F_j^k) \leqq u + 2^{1-j}. \tag{6}$$

Assume that $F=\bigcup_{j\geqq 1} M_j\notin\boldsymbol{S}_0$. Then, given $c>0$, for every j there is a nonnegative integer k such that

$$\|\mu(F_j^k)\| \geqq c, \tag{7}$$

where $\|\cdot\|$ denotes the Euclidean norm (see [3, p. 120]). Since for every i the upper or the lower variation of μ_i is bounded, say by $w_i/2$, the total variation $|\mu_i|$ of μ_i satisfies for every $M\in\boldsymbol{S}_0$ the inequality $|\mu_i|(M) \leqq |\mu_i(M)| + w_i$, especially $|\mu_i(F_j^{k+1} - F_j^k)| \leqq |\mu_i|(M_{j+k+1}) \leqq |\mu_i(M_{j+k+1})| + w_i$ and therefore for every j, k

$$\|\mu(F_j^{k+1} - F_j^k)\| \leqq w,$$

where $w = n \sup\{|\mu_i(M_j)| + w_i: i = 1, \cdots, n; j = 1, 2, \cdots\}$ is finite and independent of j and k. Also $\|\mu(F)_j^0\| = \|\mu(M_j)\| \leqq w$. For every j let $k(j)$ be the smallest nonnegative integer k satisfying (7). Then

$$c \leqq \|\mu(F_j^{k(j)})\| \leqq c + w \qquad (j = 1, 2, \cdots). \tag{8}$$

A closed and bounded subset of the E^n is compact. Thus it follows from (6) and (8) that there is an $\eta \in \pi \cap \overline{R}$, $\|\eta\| \geqq c$; c was given arbitrarily positive, in contradiction to the boundedness of $\pi \cap \overline{R}$. Hence $F \in \boldsymbol{S}_0$.

The restriction μ' of μ to the σ-ring $\boldsymbol{S}' = \{F \cap M: M \in \boldsymbol{S}\}$ is a finite vector measure. Since $M_j \in \boldsymbol{S}'$, $j = 1, 2, \cdots$, ξ is a cluster point of the range of μ'. According to (1), $\xi \in R$.

References

1. R. Borges, *Randpunkte des Wertbereichs allgemeiner Vektormasse ohne Sprünge*, Arch. Math. **16** (1965), 208–213.

2. P. R. Halmos, *The range of a vector measure*, Bull. Amer. Math. Soc. **54** (1948), 416–421.

3. ———, *Measure theory*, Van Nostrand, New York, 1950.

4. A. Liapounoff, *Sur les fonctions-vecteurs complètement additives*. Bull. Acad. Sci. URSS Sér. Math. **4** (1940), 465–478.

Universität Köln, Germany