HOMOTOPIC ARCS ARE ISOTOPIC

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In the following all spaces are simplicial complexes and all maps are piecewise linear. A simple arc α is *regularly embedded* in a compact 2-manifold M if the intersection of α with the boundary of M contains exactly the endpoints of α . Two simple arcs α and β with common endpoints p and q are *isotopic with endpoints fixed* if there exists a continuous family of homeomorphisms h_t , $0 \leq t \leq 1$, mapping M onto M such that h_0 is the identity, $h_1(\beta) = \alpha$, $h_t(p) = p$, and $h_t(q) = q$.

It is the purpose of this note to prove:

THEOREM. *Let α and β be simple arcs regularly embedded in a compact 2-manifold M and having common endpoints p and q . Let α and β intersect in only a finite number of points. If α and β are homotopic, they are isotopic.*

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Let us briefly consider the restrictions on α and β . It is not necessary that α and β be polygonal arcs nor that they intersect only a finite number of times. The theorem is, however, false if the arcs are not required to be regularly embedded. Figure 1 illustrates that homotopic arcs with common endpoints may fail to be isotopic although each lies wholly in the interior of the manifold.

Throughout the remainder of this note α , β , p , q , and M will be as in the theorem. T will be tubular neighborhood of α ; that is T is a subset of M homeomorphic to $(-1, 1) \times \alpha$ under a homeomorphism ϕ such that $\phi(0 \times \alpha) = \alpha$ and $\phi(1 - 1, 1) \times \{p, q\}$ are contained in the boundary of M . The components of $T - \alpha$ will be denoted T_1 and T_2 . A *subloop* of α and β , (*subloop*), is a simple closed curve formed by a segment of α and a segment of β . We shall only consider subloops which are determined by intersection points consecutive with respect to a parameterization of β . An intersection point x of α and β will be called a *crossing point* of α and β if either x is an endpoint of α and β or α crosses from one side of β to the other at x . Note that if α crosses β at x , then β also crosses α at x .

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FIGURE 1. Arcs in an annulus.

The theorem will be proved by showing:

(1) Intersection points which are not crossing points can be removed isotopically without introducing new intersection points. In fact, if x is an intersection point of α and β which is not a crossing point, it is possible to construct an isotopy which is the identity on the complement of a neighborhood N_x of x pulling β away from α at x . Choosing N_x sufficiently small prevents the isotopy from introducing new intersection points.

(2) When α and β intersect in more than two points, each intersection being a crossing point, an isotopy can be found transforming at least one crossing point into an intersection point, not a crossing point. This isotopy will not introduce new intersection points nor will it move p or q .

(3) When α and β intersect exactly in two crossing points, i.e., their endpoints, an isotopy exists which satisfies the conditions of the theorem.

Isotopies of forms (1) and (2) are found inductively until only two intersection points remain. The isotopy is finished using (3). Thus the proof of the theorem has been reduced to proving statements (1), (2), and (3).

The proof of statements (2) and (3) falls naturally into the same two sections. In the first a special set of generators for $\pi_1(M, p)$ will be chosen. These generators will be used to show that α and β have a trivial subloop, that is, a subloop homotopic to a point. In the

second section, it will be shown that a trivial subloop bounds a disk and that the isotopy required in either statement can be accomplished by deforming β across this disk.

1. Construction of the generators. It will be convenient to adopt a standard parameterization of α and β at this point. Let $\alpha^\rightarrow(t)$, $\beta^\rightarrow(t)$ be fixed parameterizations of α and β such that $\alpha^\rightarrow(0) = \beta^\rightarrow(0) = p$ and $\alpha^\rightarrow(1) = \beta^\rightarrow(1) = q$. Let $\alpha^\leftarrow(t) = \alpha^\rightarrow(1-t)$ and $\beta^\leftarrow(t) = \beta^\rightarrow(1-t)$. Let the symbol $\beta^\rightarrow \alpha^\leftarrow$ be the closed path obtained by following $\beta^\rightarrow$ with $\alpha^\leftarrow$. Let λ_i be the subloop of α and β determined by $\beta^\rightarrow(t_i)$ and $\beta^\rightarrow(t_{i+1})$; these being the i th and $i+1$ th intersection points along $\beta^\rightarrow$. Let l_i be the loop obtained by tracing α from p to $\beta^\rightarrow(t_i)$, tracing β from $\beta^\rightarrow(t_i)$ to $\beta^\rightarrow(t_{i+1})$, and finally tracing α from $\beta(t_{i+1})$ to p . Note that l_i is freely homotopic to λ_i . We shall denote by $[l_i]$ the element of $\pi_1(M, p)$ represented by l_i . We remark that $\prod_i [l_i] = [\beta^\rightarrow \alpha^\leftarrow]$. Now we shall say that l_i ends in T_j for $j=1, 2$, if for t less than but sufficiently near t_{i+1} , $\beta^\rightarrow(t) \in T_j$. Likewise l_i begins in T_j if for t greater than but sufficiently near t_i , $\beta^\rightarrow(t) \in T_j$.

Let us assume that M is separated by α . Let x_1 and x_2 be the closures of the components of $M - \alpha$. Let $\{V_1 \cdots V_n\}$ be a system of loops which lie in x_1 such that $\{[V_1], \cdots, [V_n]\}$ is a system of free generators for $\pi_1(x_1, p)$ and let $\{V_{j+1}, \cdots, V_n\}$ be the same in x_2 . Since x_i for $i=1, 2$, is a 2-manifold with boundary, $\pi_1(x_i, p)$ is a free group. Applying van Kampen's theorem, we see that $\{[V_1] \cdots [V_n]\}$ is a free system of generators for $\pi_1(M, p)$ since $\pi_1(x_1 \cap x_2, p)$ is a trivial group. In case α separates M , this will be our free system of generators for $\pi_1(M, p)$.

Let us assume that $M - \alpha$ is connected. Let x_1 be $M - T_2$. Let μ be a path which lies in x_1 and irreducibly joins the two components of the intersection of x_1 and the closure of T_2 . Let x_2 denote the union of μ and the closure of T_2 . By van Kampen's theorem the union of a free set of generators for $\pi_1(x_1, p)$ and a free set of generators for $\pi_1(x_2, p)$ is a free set of generators for $\pi_1(M, p)$. Note that $\pi_1(x_2, p)$ is a free group since x_2 has the homotopy type of a circle. Let V_1 be a simple loop originating at p , traversing α from p to the intersection of α and μ next traversing μ , and finally passing through T_2 to p . Then $[V_1]$ generates $\pi_1(x_2, p)$. Choose a set of simple loops $\{V_2 \cdots V_n\}$ embedded in x_1 such that $\{[V_2] \cdots [V_n]\}$ is a free set of generators for $\pi_1(x_1, p)$. Now $\{[V_1], \cdots, [V_n]\}$ is a free set of generators for $\pi_1(M, p)$. This will be our free set of generators for $\pi_1(M, p)$ if $M - \alpha$ is connected.

LEMMA 1. *Let $M - \alpha$ be connected. Assume that λ_k is a nontrivial sub-*

loop of α and β . The loop l_k ends in T_2 if and only if the reduced word for $[l_k]$ ends with $[V_1]$. The loop l_k begins in T_2 if and only if the reduced word $[l_k]$ begins with $[V_1]^{-1}$.

PROOF. Assume that l_k begins in T_1 and ends in T_2 . The loop $[l_k][V_1]^{-1}$ is homotopic to a loop in x_1 . Thus $[l_k][V_1]^{-1}$ is a word in the generators $[V_2], \dots, [V_n]$, and the reduced word for $[l_k]$ ends in $[V_1]$.

The remaining three cases in the proof are similar.

LEMMA 2. Assume that each intersection point of α and β is a crossing point. Then α and β have a trivial subloop.

PROOF. Assume to the contrary that no $[l_i] = 1$. Since $[\beta^+ \alpha^-] = 1$, there exists a k such that formal cancellation occurs between the words for $[l_k]$ and $[l_{k+1}]$. It follows from the definition of crossing point that if l_k ends in T_2 , l_{k+1} begins in T_1 and conversely.

Case 1. Assume $M - \alpha$ is connected. Assume that l_k ends in T_2 . By Lemma 1, $[l_k]$ ends in $[V_1]$ and $[l_{k+1}]$ does not begin in $[V_1]^{-1}$. This contradicts our assumption of formal cancellation. The proof is similar if l_k ends in T_1 .

Case 2. Assume α separates M . A subloop lies in x_1 or x_2 . If l_k lies in x_1 , then l_{k+1} lies in x_2 and $[l_k]$ and $[l_{k+1}]$ are words in $[V_1] \cdots [V_j]$ and $[V_{j+1}] \cdots [V_n]$ respectively. But then there can be no formal

DIAGRAM 1

cancellation between $[l_k]$ and $[l_{k+1}]$. We have a contradiction. The proof is similar if l_k lies in x_2 .

2. Construction of the isotopy.

LEMMA 3. *Let λ be a simple closed curve embedded in M and homotopic to a point. Then λ bounds a disk, proved in less generality in [2].*

PROOF. Assume that λ is interior to M . Since λ is homotopic to a point, it is homologous to zero. Thus there exists a submanifold C_1 of M bounded by λ . Let C_2 be the closure of the complement of C_1 . Assume that neither C_1 nor C_2 is a disk, then it is easy to see that λ is not homotopic to a point in either C_1 or C_2 . It follows that f_1 and f_2 (induced by inclusion) in Diagram 1 are one-one.

According to a lemma of Brown and Crowell [1], f_3 and f_4 (induced by inclusion) are also 1-1. But then λ is not homotopic to a point in $\pi_1(M)$. It follows that C_1 or C_2 must have been a disk.

If λ is not interior to M , construct a manifold M_1 by joining an annulus to each component of the boundary of M whose intersection with λ is not void. Then λ is interior to M_1 and, by the above argument, λ bounds a disk in M_1 . This disk is contained in M since it does not meet the boundary of M_1 .

FIGURE 2.

Assume that α and β have more than two intersection points, all being crossing points. Let λ_k be a trivial subloop of α and β . By Lemma 3, λ_k bounds a disk, D . We observe that all points of α contained in D lie on its boundary. Let P be disk which is a neighborhood of $D - \{\beta(t_k), \beta(t_{k+1})\}$. Let $\beta(t_k)$ and $\beta(t_{k+1})$ be on the boundary of P . Choose P so small that the component of $P - D$ with $\alpha \cap \lambda_k$ on its boundary contains no points of α . Further the component of $P - D$ with $\beta \cap \lambda_k$ on its boundary should not meet β . Choose an isotopy h_t leaving points outside of P fixed and moving $\beta \cap \lambda_k$ across D and beyond α . See Figure 2. Then $h_1(\beta(t)) = \beta(t)$ or $h_1(\beta(t))$ lies in the component of $P - D$ containing no points of α . Therefore we have introduced no new intersection points. Also either $\beta(t_k)$ or $\beta(t_{k+1})$ is no longer a crossing point. Statement (2) follows.

Assume that α and β intersect in exactly two points, p and q . Let P and D be as above and choose an isotopy h_t leaving points outside of P fixed and moving β across D and onto α . Statement (3) follows.

BIBLIOGRAPHY

1. E. M. Brown and R. H. Crowell, *The augmentation subgroup of a link*, J. Math. Mech. **15** (1966).
2. H. I. Levine, *Homotopic curves on surfaces*, Proc. Amer. Math. Soc. **14** (1963), 986.

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