ON A CLASS OF MEROMORPHIC FUNCTIONS

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In his paper [1] F. Gross considers functions $f(z)$ and $g(z)$ meromorphic in the plane and satisfying

$$f^n + g^n = 1, \tag{1}$$

where n is a fixed integer. For $n=2$ he shows that all meromorphic solutions of (1) are of the form

$$f = \frac{2\beta}{1+\beta^2}, \qquad g = \frac{1-\beta^2}{1+\beta^2},$$

where β is meromorphic. In this case one may even obtain entire solutions, e.g. $f=\sin z$, $g=\cos z$, $\beta=\tan(z/2)$. Gross also shows that for $n>2$ there are no entire solutions of (1), while for $n>3$ there are no meromorphic solutions.

Now the equation $w^3+z^3=1$ defines an algebraic function whose Riemann surface has genus 1, and there is accordingly a uniformization by elliptic functions. If $\wp(z)$ is the Weierstrass elliptic function with periods ω_1, ω_2 satisfying

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3, \qquad g_2, g_3 \text{ constants},$$

then (cf. [2, p. 227]) ω_1 and ω_2 may be chosen so that

$$g_2 = 0, \qquad g_3 = 1.$$

With this $\wp(z)$ we find that

$$\begin{aligned} f(z) &= \left\{\frac{1}{2} + \frac{\wp'(z)}{(12)^{1/2}}\right\} \Big/ \wp(z), \\ g(z) &= \left\{\frac{1}{2} - \frac{\wp'(z)}{(12)^{1/2}}\right\} \Big/ \wp(z), \end{aligned} \tag{2}$$

satisfy

$$f^3 + g^3 = 1. \tag{3}$$

The formulas (2) differ from the analogous formulae in [1], which seem to contain an error.

With the aid of the functions in (2) one may verify a conjecture made by F. Gross in [1], viz. that meromorphic solutions of (3) are

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necessarily elliptic functions of entire functions. We shall prove

THEOREM 1. *Any functions* $F(z)$, $G(z)$, *which are meromorphic in the plane and satisfy*

$$F^3 + G^3 = 1, \tag{4}$$

have the form

$$F = f(h(z)), \qquad G = \eta g(h(z)) = \eta f(-h(z)) = f(-\eta^2 h(z)), \tag{5}$$

where f and g are the elliptic functions in (2), $h(z)$ *is an entire function of z and* η *is a cube-root of unity.*

PROOF. Write $\rho = \exp(2\pi i/3)$. If F and G are meromorphic solutions of (4), then since $F = (1-G^3)^{1/3}$ is single-valued, it follows that the multiplicity of any solution z of $G(z) = \rho$, ρ^2 or 1, is a multiple of 3.

We shall need to discuss the singularities of the inverse function $f_{-1}(w)$ of the $f(z)$ in (2). Since $f(z)$ is a doubly periodic function it has neither finite nor infinite asymptotic values and hence, by Iversen's theorem, all the singularities of $f_{-1}(w)$ are algebraic. We prove that these singularities all lie over $w = \rho$, ρ^2 or 1. First note that $\wp(z)$ has double poles at the points $m\omega_1 + n\omega_2$, m, n integral, and so $f(z)$ has single poles at these points. $\wp(z)$ has poles nowhere else, so that the other poles of $f(z)$ are at the zeros of $\wp(z)$ and hence there are two simple ones in each period parallelogram, since $\wp(z)$ takes each value twice in such a parallelogram, while $\wp(z) = 0$ implies

$$(\wp')^2 = 4\wp^3 - 1 = -1 \neq 0.$$

Thus altogether $f(z)$ has three simple poles in each period parallelogram S, while by differentiation $f'(z)$ has three double poles and thus f and f' take each value three or six times respectively, in S. Now $f^3 + g^3 = 1$, so that by the first remarks of this proof $f = \rho$, ρ^2 and 1 at least triply at each solution. Thus f takes each value ρ, ρ^2 and 1 precisely at one point in S, the derivative f' having a double zero at each of these points and at no other points of S. Thus all singularities of $f_{-1}(w)$ lie over $w = \rho$, ρ^2 and 1. In particular $w = \infty$ is a regular point of each branch of $f_{-1}(w)$.

We return to the consideration of F, G satisfying (4), and in the neighborhood of any value z_0, such that $w = F(z_0) \neq \rho$, ρ^2, 1, we take any branch of $f_{-1}(w)$ and form the regular function element

$$h(z) = f_{-1}(F(z)).$$

Now $h(z)$ may be continued analytically along any curve γ in the plane without restriction. Obviously the continuation can only fail when γ reaches a point z_1 such that $w_1=F(z_1)=\rho$, ρ^2 or 1. Denote by γ_1 the arc of γ between z_0 and z_1, exclusive of the end point z_1. Then $h(z)$ is regular along γ_1 and for each point z on γ_1 there is a branch of f_{-1} such that $f_{-1}(F(z))=h(z)$. Now z_1 is a $3k$-fold solution of $F(z_1)=w_1$, so $F(z)=w_1+\{\phi(z)\}^3$, where $\phi(z)$ is a regular function in the neighborhood N: $|z-z_1|<\delta$, $\delta>0$, and satisfies $\phi(z_1)=0$. We may suppose δ chosen so small that for z in N we have $|F(z)-w_1|<1$. Now in the neighborhood M: $|w-w_1|<1$, the only branch points of $f_{-1}(w)$ lie over $w=w_1$. For some z in $\gamma_1\cap N$ we form $w=F(z)$ in M and choose the branch $f_{-1}(w)$ such that $f_{-1}(F(z))=h(z)$. We note that for neighboring values z we obtain the same branch $f_{-1}(w)$, which indeed has an expansion

$$f_{-1}(w) = \lambda + P((w - w_1)^{1/3}), \qquad |w - w_1| < (3)^{1/2},$$

where λ is a constant and $P(t)$ is a convergent power series in t. Thus we must have for all z in $\gamma_1\cap N$, using $F=w_1+\phi^3$, an expression

$$h(z) = \lambda + P(\mu\phi),$$

where μ is a fixed 3rd root of unity and ϕ is regular in N. This expression gives a regular continuation of $h(z)$ over the value z_1. Thus we have verified that $h(z)$ can be continued throughout the plane to give (by the monodromy theorem) a function regular in the plane i.e. an entire function.

We now have $F(z)=f(h(z))$ and

$$F^3 + G^3 = 1,$$
$$f^3 + g^3 = 1,$$
$$f(h)^3 + g(h)^3 = 1 = F^3 + g(h)^3.$$

Hence $G^3=g(h)^3$, $G=\eta g(h)$, where η is (since G, $g(h)$ are regular) a fixed third root of unity. Since $\wp$ is even and $\wp'$ is odd we have $f(-z)=g(z)$ and G can also be written $\eta f(-h)$.

We remark finally that (cf. [2, p. 168]) $\wp(z)$ has an expansion

$$\wp(z) = \frac{1}{z^2} + \sum_{n=2}^{\infty} C_n z^{2n-2},$$

where

$$(n-3)(2n+1)C_n = 3(C_2C_{n-2} + C_3C_{n-3} + \cdots + C_{n-2}C_2), \qquad n = 4, 5, 6, \cdots,$$

$$C_2 = \frac{1}{20} g_2, \qquad C_3 = \frac{1}{28} g_3.$$

Since $g_2=0$, $g_3=1$, it is easy to prove inductively that $C_n=0$ unless $n\equiv 0$ modulo 3. Substitution in (2) shows that $zf(z)$ is a function of z^3, so that

$$f(\eta z) = \eta^2 f(z), \quad \eta^3 = 1.$$

This shows that $f(-\eta^2 h(z)) = \eta f(-h(z))$, and the proof of the equivalence of the various expressions for G in (5) is complete.

References

1. F. Gross, *On the equation $f^n+g^n=1$*, Bull. Amer. Math. Soc. **72** (1966), 86–88.
2. A. Hurwitz and R. Courant, *Funktionentheorie*, 4 Auflage, Springer, Berlin, 1964.

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ON THE BOUNDARY BEHAVIOR OF FUNCTIONS MEROMORPHIC IN THE UNIT DISK

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1. **Introduction.** Let $f(z)$ be meromorphic in D: $\{|z|<1\}$, and suppose that the values assumed by $f(z)$ in D lie in a domain G whose boundary Γ has positive logarithmic capacity. Then $f(z)$ is of bounded characteristic in D and has finite radial limits $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ at almost all points $e^{i\theta}$ on C: $\{|z|=1\}$. (For this and more general theory of meromorphic functions, see [**4**, pp. 208 ff.].) The class of functions satisfying these conditions and having the additional property that $f(e^{i\theta})$ belongs to Γ almost everywhere on C has been studied by O. Lehto [**3**] and D. A. Storvick [**6**], who called it *class* (L).

If A is a sequence of points in D satisfying $\sum_{a\in A}(1-|a|)<\infty$, the Blaschke product with respect to A in D is the function $B(z;A) = \prod_{a\in A}[|a|(a-z)/a(1-\bar{a}z)]$. The present note arises from a suggestion by Professor Storvick that the following theorem, established in [**1**], be extended to functions in class (L). Here we denote by A' the derived set of A.

Theorem 1. *Let E be a set on C. A necessary and sufficient condition*

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