THE MAXIMAL GCR IDEAL IN AN AW*-ALGEBRA

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1. **Introduction.** Kaplansky [**3**] introduced the notion of AW*-algebras to study algebraically certain properties of von Neumann algebras. Each AW*-algebra $\mathfrak{A}$ can be written as the product of a discrete part $\mathfrak{A}P$ and a continuous part $\mathfrak{A}(1-P)$, where P is a projection in the center Z of $\mathfrak{A}$. The discrete part $\mathfrak{A}P$ is characterized by the fact that each nonzero projection majorizes an abelian projection E, i.e. a nonzero projection such that $E\mathfrak{A}E = ZE$; the continuous part $\mathfrak{A}(1-P)$ contains no abelian projections at all.

In a C^*-algebra a two-sided closed ideal I is said to be a CCR ideal if and only if every irreducible representation Φ of the algebra on a Hilbert space H such that $\Phi(I)\neq(0)$ maps I onto the ideal of completely continuous operators on H [**2**]. An ideal I in the C^*-algebra is a GCR ideal if and only if there are ideals $(I_\rho)_{0\leq\rho\leq\alpha}$ (ρ an ordinal) in the algebra such that

(1) $I_\beta < I_\gamma$ for $\beta<\gamma$;
(2) Closure $[\bigcup\{I_\rho \mid \rho<\beta\}] = I_\beta$, if β is a limit ordinal;
(3) $I_{\rho+1}/I_\rho$ is a CCR ideal in the algebra reduced mod I_ρ; and
(4) $I_0=(0)$, $I_\alpha = I$.

In every C^*-algebra there is a maximal GCR ideal I such that the algebra mod I contains no GCR ideals.

In an AW*-algebra $\mathfrak{A}$ the maximal GCR ideal of $\mathfrak{A}$ is contained in the discrete part of $\mathfrak{A}$. For this reason we consider only discrete (Type I) AW*-algebras. The $*$-subalgebra I_a of $\mathfrak{A}$ generated by the abelian projections of $\mathfrak{A}$ is an ideal in $\mathfrak{A}$ [**4**]. In this paper we show that not only is I_a the maximal GCR in $\mathfrak{A}$ but also that I_a is a CCR ideal and that $\mathfrak{A}/J$ has no CCR ideals if J is an ideal containing I_a.

In a Type I AW*-algebra $\mathfrak{A}$, a nonzero projection P is said to be finite if it is equivalent to no proper subprojection. An AW*-algebra is finite if every nonzero projection is finite. A nonzero projection P in the center of $\mathfrak{A}$ is said to be properly infinite if P is not finite and if P majorizes no proper finite projection in the center of $\mathfrak{A}$. In every Type I algebra $\mathfrak{A}$ we may find a central projection P such that $\mathfrak{A}P$ is a finite algebra and such that $1-P$ is a properly infinite projection. We characterize a finite Type I algebra in terms of the hull of the ideal of I_a in the strong structure space $M(\mathfrak{A})$ of maximal ideals of $\mathfrak{A}$. For part of this characterization we use a theorem of Fred B. Wright [**6**]. We also show that the interior of the hull of I_a corre-

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sponds to the properly infinite projections. Finally we show that the structure space of I_a (i.e. the set of all primitive ideals of I_a taken with the hull-kernel topology) is Hausdorff.

2. **The maximal** GCR **ideal in a Type I algebra.** We shall make repeated use of the following lemma.

LEMMA. *If I is a closed two-sided ideal in an* AW*-*algebra* $\mathfrak{A}$, *then I is the closure of the* *-*subalgebra of* $\mathfrak{A}$ *generated by the projections in I.*

THEOREM 1. *If* $\mathfrak{A}$ *is an* AW*-*algebra of Type* I, *the maximal* GCR *ideal in* $\mathfrak{A}$ *is the closed* *-*subalgebra* I_a *of* $\mathfrak{A}$ *generated by the abelian projections. This ideal is, in fact, a* CCR *ideal.*

PROOF. We first show that the ideal I_a of abelian projections is a CCR ideal. Let Ψ be an irreducible representation of I_a on a Hilbert space H with kernel Q. There is an irreducible representation Φ of $\mathfrak{A}$ on H such that $\Phi | I_a = \Psi$. If $\mathfrak{Z}$ is the center of $\mathfrak{A}$, there is a maximal ideal ζ in $\mathfrak{Z}$ such that $\mathfrak{Z} \cap P = \zeta$ where P is the kernel of Φ. Let E be an abelian projection of $\mathfrak{A}$ such that $\Psi(E) \neq 0$. Since $E\mathfrak{A}E = \mathfrak{Z}E$, we have that $\Phi(E)\Phi(A)\Phi(E) = \alpha\Phi(E)$, ($\alpha$, a scalar), for each A in $\mathfrak{A}$. If $EAE = BE$, $B \in Z$, $\alpha = \hat{B}(\zeta)$, where $\hat{B}$ is the image of B under the Gelfand isomorphism. So $\Phi(E) = \Psi(E)$ is a one-dimensional projection on H. Because $\Psi(I_a)$ is irreducible on H and because $\Psi(I_a)$ is generated by one-dimensional projections, the *-algebra $\Psi(I_a)$ is exactly the algebra of completely continuous operators on H. This completes the proof of the fact that I_a is a CCR ideal.

We now prove that I_a is the maximal GCR ideal by assuming that there is a GCR ideal J properly containing I_a and then obtaining a contradiction. We have that $K = J/I_a$ is a GCR ideal in $\mathfrak{A}/I_a$. There is a composition series $\{K_\rho | 0 \leqq \rho \leqq \beta\}$ of ideals in J/I_a. We may assume $K_1 \neq 0$. Let I be the complete inverse image of I under the canonical map $\mathfrak{A}$ onto $\mathfrak{A}/I_a$; we have K_1 is isomorphic to I/I_a and thus the ideal I has the property that I/I_a is a CCR ideal.

Let E be a projection in I but not in I_a. Since $E\mathfrak{A}E$ is an AW*-algebra of Type I, we may find an orthogonal net $\{P_i\}$ of projections in the center of $\mathfrak{A}$ such that each algebra $E\mathfrak{A}E \cdot P_i$ is homogeneous. There is for each P_i a net $\{E_{ik}\}_k$ of orthogonal equivalent abelian projections such that $\sum_k E_{ik} = P_i E$. Assume first that there is an index i such that $P_i E \notin I_a$. Then the number of abelian projections in the sum $\sum_k E_{ik}$ must be infinite. Now there is a representation Φ of $\mathfrak{A}$ onto a Hilbert space with the following properties: (1) the kernel of Φ contains I_a; (2) $\Phi(P_i E) \neq 0$; and (3) $\Phi(I)$ is the algebra of completely continuous operators on H. Since $\Phi(P_i E) \neq 0$ is a member of the algebra of completely continuous operators, $\Phi(P_iE)$ is a finite dimensional projection on H; suppose $\dim \Phi(P_iE)=n$. There are $2n$ equipotent disjoint sets $S_1, S_2, \cdots, S_{2n}$ of the indexing set of the k's in $\{E_{ik}\}_k$ whose union is the indexing set. Then if $E_j=\sum\{E_{ik} | k\in S_j\}$ $(1\leqq j\leqq 2n)$ we have that $E_1, E_2, \cdots, E_{2n}$ is a set of orthogonal equivalent projections of sum EP_i. Thus, $\Phi(EP_i)$ is the sum of $2n$ nonzero orthogonal projections. This is a contradiction. Hence for each i we have that EP_i lies in I_a.

We now prove that each EP_i is the sum of a finite number $n(i)$ of orthogonal equivalent abelian projections $\{E_{ik}\}_k$. Indeed, let ζ be a maximal ideal in the center $\mathcal{Z}$ of $\mathcal{A}$ such that EP_i does not lie in the two-sided ideal $[\zeta]$ given by

$$[\zeta] = \text{closure}\left\{ \sum_{j=1}^{n} A_jB_j \,\middle|\, A_j \in \mathcal{A},\ B_j \in \zeta, \right.$$
$$\left. \text{for all } j = 1, \cdots, n \text{ and for all } n = 1, 2, \cdots \right\}.$$

There is a Hilbert space $H(\zeta)$ and a representation Ψ_ζ of $\mathcal{A}$ on $H(\zeta)$ such that the kernel of Ψ_ζ is equal to $[\zeta]$ and such that $\Psi_\zeta(I_a)$ is the ideal of completely continuous operators on $H(\zeta)$. The nonzero projection $\Psi_\zeta(P_iE)$ has finite dimension n. If the indexing set S of the k's in $\{E_{ik}\}_k$ were infinite, then we could write $S=\bigcup\{S_j | 1\leqq j\leqq 2n\}$ where $S_1, S_2, \cdots, S_{2n}$ are equipotent disjoint subsets of S. If we set $E_j=\sum\{E_k | K\in \ddot{S}_j\}$ $(1\leqq j\leqq 2n)$, the projections $E_1, E_2, \cdots, E_{2n}$ are orthogonal and equivalent with sum EP_i. Since each $\Psi_\zeta(E_j)\neq 0$, the projection $\Psi_\zeta(P_iE)$ has dimension greater than n. This is a contradiction. So the number $n(i)$ of k in the indexing set is finite. If $E_1, E_2, \cdots, E_m$ and $F_1, F_2, \cdots, F_n$ are two sets of equivalent orthogonal abelian projections such that

$$\sum\{E_j \mid 1 \leqq j \leqq m\} = \sum\{F_j \mid 1 \leqq j \leqq n\},$$

we have $m=n$; thus, the number $n(i)$ is unique.

For each integer n we let $S(n)=\{i | n(i)=n\}$. We show that for each integer N there is an integer $n>N$ such that $S(n)$ is nonvoid. If, on the contrary, $S(n)$ is void for $n>N$, we have that E is a member of I_a. Indeed, $E=F_1+F_2+\cdots+F_N$, where the nonzero F_k are orthogonal abelian projections given by $F_k=\sum_i E_{ik}$ $(1\leqq k\leqq N)$. Each nonzero F_k is abelian because it is the sum of disjoint abelian projections. So for each integer N there is an integer $n>N$ such that $S(n)$ is nonvoid.

We shall obtain a contradiction now by showing that for each positive integer m there are m orthogonal equivalent projections $E_1, E_2, \cdots, E_m$ such that $E \geqq E_1 + E_2 + \cdots + E_m$ and

$E-(E_1+E_2+\cdots+E_m)\in I_a$. This shows that under any representation Φ of $\mathfrak{A}$ on a Hilbert space such that the kernel of Φ contains I_a it is impossible for $\Phi(E)$ to be a completely continuous operator. Thus, I/I_a is not a CCR ideal in $\mathfrak{A}/I_a$. For each n such that $S(n)$ is nonempty we define the projections

$$G_{nk}=\sum\{E_{ik}\mid i\in S(n)\},\qquad (1\leqq k\leqq n).$$

Since $G_{n1}, G_{r2}, \cdots, G_{nn}$ are the sums of orthogonal equivalent projections, we have that the $G_{n1}, G_{n2}, \cdots, G_{nn}$ are orthogonal and equivalent. Also each G_{nk} is the sum of disjoint abelian projections; thus, each G_{nk} is abelian. For each n let $n=r(n)m+j(n)$, where $0\leqq j(n)<m$. Let

$$F_{nj}=\sum\{G_{nl}\mid (j-1)r(n)+j(n)+1\leqq l\leqq jr(n)+j(n)\},\ (1\leqq j\leqq m).$$

We have that $F_{n1}, F_{n2}, \cdots, F_{nm}$ are mutually orthogonal equivalent projections provided $n\geqq m$. For $n<m$ define $F_{nj}=0$ $(1\leqq j\leqq m)$. Since $S(n)$ is nonvoid for some $n\geqq m$, we have that there is an integer n with $F_{n1}\neq 0$. We now remark that G_{nk} and G_{pj} are disjoint for $n\neq p$. Indeed, let $P_n=\sum\{P_i\mid i\in S(n)\}$ and $P_p=\sum\{P_i\mid i\in S(p)\}$. Since $S(n)\cap S(p)$ is void, the projections P_n and P_p are orthogonal. However, $P_n\geqq G_{nk}$ and $P_p\geqq G_{pj}$; this gives the desired result.

Now we let $F_1, F_2, \cdots, F_{m-1}$ be projections given by

$$F_j=\sum\{\delta_{nj}G_{nj}\mid 1\leqq n<\infty\},\qquad (1\leqq j\leqq m-1),$$

where $\delta_{nj}=1$ if $j\leqq j(n)$ and $\delta_{nj}=0$ if $j(n)<j$. The F_j being the sum of disjoint abelian projections is abelian. We let

$$E_k=\sum\{F_{nk}\mid 1\leqq n<\infty\},\qquad (1\leqq k\leqq m).$$

The projections $E_1, E_2, \cdots, E_m$ are mutually orthogonal equivalent nonzero projections. We have

$$E=F_1+F_2+\cdots+F_{m-1}+E_1+E_2+\cdots+E_m$$

as desired. Q.E.D.

THEOREM 2. *Let $\mathfrak{A}$ be an* AW*-*algebra of Type* I *and let I_a be the *-subalgebra of $\mathfrak{A}$ generated by the abelian projections of $\mathfrak{A}$. If I is an ideal in $\mathfrak{A}$ such that $I\supset I_a$, then $\mathfrak{A}/I$ has no* CCR *ideals.*

PROOF. Let us assume that J is an ideal in $\mathfrak{A}$ such that $J\supset I$ and J/I is a CCR ideal. We shall obtain a contradiction. Let E be a projection in J but not in I. Since $E\mathfrak{A}E$ is a Type I AW*-algebra, there is a net $\{P_i\}$ of orthogonal projections in the center of $\mathfrak{A}$ such that (1) $\sum_i EP_i=E$ and (2) $E\mathfrak{A}E\cdot P_i$ is homogeneous for each i. For each

i let $\{E_{ik} \mid k \in T(i)\}$ be an orthogonal equivalent set of abelian projections of least upper bound EP_i.

There is an irreducible representation Φ of $\mathfrak{A}$ on a Hilbert space H such that (1) the kernel K of Φ contains I; (2) $\Phi(J)$ is the algebra of completely continuous operators on H; and (3) $\Phi(E) \neq 0$. We shall show that for any positive integer m there are orthogonal equivalent projections $E_1, E_2, \cdots, E_m$ such that $E \geqq E_1 + E_2 + \cdots + E_m$ and $E - (E_1 + E_2 + \cdots + E_m) \in K$. This will produce a contradiction to the fact that $\Phi(E)$ is a finite dimensional projection on H. Thus, the algebra $\mathfrak{A}/I$ will have no CCR ideals.

First assume that there is an index i such that $EP_i \notin K$; then the set $T(i)$ is infinite because otherwise $EP_i \in I_a \subset K$. By the method of Theorem 1, for any positive integer m we would be able to write $E_1 + E_2 + \cdots + E_m = EP_i$ where $E_1, E_2, \cdots, E_m$ are orthogonal equivalent projections. Since $\Phi(EP_i) \neq 0$ is a completely continuous operator on H, we would have a contradiction. Therefore, for each i we have $EP_i \in K$.

Now let $|T(i)|$ denote the cardinality of the set $T(i)$. For each positive integer n there is a set $T(i)$ whose cardinality exceeds n, otherwise $E \in I_a$. Let $S_1 = \{i \mid |T(i)| < \infty\}$ and $S_2 = \{i \mid |T(i)|$ is not finite$\}$ we write, as in Theorem 1,

$$E_1' + E_2' + \cdots + E_m' + E_{m+1}' = \sum \{EP_i \mid i \in S_1\},$$

where $E_1', E_2', \cdots, E_{m+1}'$ are mutually orthogonal projections such that $E_1' \sim E_2' \sim \cdots \sim E_m'$ and $E_{m+1}' \in I_a$. We do not know in this case whether or not any of the E_k' are nonzero. For each i in S_2 we may write $T(i) = \bigcup \{T(i, j) \mid 1 \leqq j \leqq m\}$, where $T(i, j)$ $(1 \leqq j \leqq m)$ are disjoint equipotent subsets of $T(i)$. We let

$$F_{ij} = \sum \{E_{i\rho} \mid \rho \in T(i, j)\}, \quad \text{for } (1 \leqq j \leqq m).$$

Then $F_{i1}, F_{i2}, \cdots, F_{im}$ are mutually orthogonal equivalent projections. Let

$$E_j'' = \sum \{F_{ij} \mid i \in S_2\}, \qquad (1 \leqq j \leqq m).$$

The projections $E_1'', E_2'', \cdots, E_m''$ are mutually orthogonal and equivalent. For each j and k such that $1 \leqq j, k \leqq m$, we have that E_j' is orthogonal to E_k''. Thus, if we set $E_j = E_j' + E_j''$ $(1 \leqq j \leqq m)$, we have that $E_1, E_2, \cdots, E_m$ are orthogonal equivalent projections such that $E \geqq E_1 + E_2 + \cdots + E_m$ and such that $E - (E_1 + E_2 + \cdots + E_m) = E_{m+1}' \in I_a$. This completes the proof of Theorem 2.

3. **Structure space of Type I algebras.** Let $\mathfrak{A}$ be an AW*-algebra and let $\mathfrak{Z}$ be the center of $\mathfrak{A}$; let $M(\mathfrak{A})$ be the set of all maximal ideals of $\mathfrak{A}$ and let Z be the set of all maximal ideals of $\mathfrak{Z}$, i.e. the spectrum of $\mathfrak{Z}$. The set $M(\mathfrak{A})$ is given the hull-kernel topology and the set Z is given the w^*-topology when Z is identified with the set of all nonzero complex-valued homomorphism of $\mathfrak{Z}$. There is a homeomorphism of $M(\mathfrak{A})$ onto Z given by $M \rightarrow M \cap \mathfrak{Z} \{ M \in M(\mathfrak{A}) \}$. If $M \cap \mathfrak{Z} = \zeta \in Z$ where $M \in M(\mathfrak{A})$ we let $M = M(\zeta)$. Now if A is an element of $\mathfrak{Z}$, let $\hat{A}$ be the image of A under the Gelfand map in the algebra of all continuous complex-valued functions on Z. If Q is a projection in $\mathfrak{Z}$, the set $\mathfrak{A}Q = I$ is a closed two-sided ideal in $\mathfrak{A}$. If $h(I)$, the hull of I, is the set $h(I) = \{ M \in M(\mathfrak{A}) \mid M \supset I \}$, we have that $M(\mathfrak{A}) - h(I) = \{ M(\zeta) \in M(\mathfrak{A}) \mid \hat{Q}(\zeta) = 1 \}$; we also have that $M \cap I = MQ$ for all M in $M(\mathfrak{A}) - h(I)$.

THEOREM 3. *Let $\mathfrak{A}$ be an* AW*-*algebra of Type* I *and let I_a be the *-subalgebra of $\mathfrak{A}$ generated by the abelian projections. Then $\mathfrak{A}$ is finite if and only if $h(I_a)$ is nowhere dense in $M(\mathfrak{A})$.*

PROOF. Let $\mathfrak{A}$ be finite. Fred B. Wright [6] has proved that the algebras $\mathfrak{A}/M$ $\{ M \in M(\mathfrak{A}) \}$ are finite Type I AW*-algebras except possibly on a nowhere dense set $N \subset M(\mathfrak{A})$. The set N is void if and only if the number of homogeneous summands of $\mathfrak{A}$ is finite. If N is nonvoid, then $M \in N$ if and only if $\mathfrak{A}/M$ is an AW*-factor of Type II_1. We immediately see that $h(I_a) = N$ and thus that $h(I_a)$ is nowhere dense in $\mathfrak{A}$.

Now assume that $\mathfrak{A}$ is not finite. We shall show that $h(I_a)$ contains a nonvoid open set. Let $\{P_i\}$ be a net of orthogonal projections in $\mathfrak{Z}$ such that $\sum_i P_i$ is the identity operator and such that $\mathfrak{A}P_i$ is homogeneous for each i. We may write $\sum_i E_{ik} = P_i$ where $\{E_{ik}\}_k$ is a net of orthogonal equivalent abelian projections of $\mathfrak{A}$. Since $\mathfrak{A}$ is not finite, there is a P_i such that the cardinality of the set of $\{E_{ik}\}_k$ is not finite. Let $P_i = Q$. We have that $h(\mathfrak{A}Q) \neq M(\mathfrak{A})$ since Q is not the zero projection. We shall prove the nonempty open set $M(\mathfrak{A}) - h(\mathfrak{A}Q)$ is contained in $h(I_a)$. If $M \in M(\mathfrak{A}) - h(\mathfrak{A}Q)$, we have $M = M(\zeta)$ for some ζ such that $\hat{Q}(\zeta) = 1$. There is a representation Ψ of $\mathfrak{A}$ onto a Hilbert space H such that the kernel $[\zeta]$ of Ψ is the closure of $\{ \sum_k \{ A_k B_k \mid A_k \in \mathfrak{A}, B_k \in \zeta \ (1 \leq k \leq n) \} \ n = 1, 2, \cdots \}$ and such that $\Psi(I_a)$ is the algebra of completely continuous operators on H. Using the representation Ψ, we prove the closed two-sided ideal J generated by I_a and $[\zeta]$ is proper in $\mathfrak{A}$. On the contrary, if J contained the identity of $\mathfrak{A}$, there is an element A in I_a and an element B in $[\zeta]$ such that $\|A+B-1\| < 1$. This means that $A+B$ has an inverse in $\mathfrak{A}$ and hence that

$\Psi(A+B)=\Psi(A)$ has an inverse in $\Psi(\mathfrak{A})$. However, $\Psi(A)$ is a completely continuous operator and $\Psi(A)$ will have no inverse if H is infinite dimensional. For any integer $n>0$ we may write Q as the sum of n orthogonal equivalent projections since Q is the sum of an infinite set of orthogonal equivalent abelian projections. For the integer $n>0$ the dimension of H is not less than n because $\Psi(Q)\neq 0$. So the dimension of H is infinite. We are forced to conclude that J is a proper ideal. This means J is contained in a maximal ideal. Since there is one and only one maximal ideal containing $[\zeta]$, we have $J\subset M(\zeta)$ and so $I_a\subset M(\zeta)$. This completes the proof.

The next theorem characterizes the interior of $h(I_a)$ in terms of the properly infinite central projections.

THEOREM. *Let $\mathfrak{A}$ be an* AW**-algebra of Type* I *with center $\mathfrak{Z}$. A projection P in $\mathfrak{Z}$ is properly infinite if and only if $M(\mathfrak{A})-h(\mathfrak{A}P)\subset h(I_a)$. In particular, the identity* 1 *is properly infinite if and only if $M(\mathfrak{A})=h(I_a)$.*

PROOF. Let P be a properly infinite central projection and let $M=M(\zeta)\in M(\mathfrak{A})-h(\mathfrak{A}P)$. We obtain a contradiction by assuming that $M\not\supset I_a$. Let $\{P_i\}$ be a net of orthogonal central projections with least upper bound P such that for each i the algebra $\mathfrak{A}P_i$ is homogeneous. For each i let $\{E_{ik}\mid k\in T(i)\}$ be a net of orthogonal equivalent abelian projections of least upper bound P_i. Since each P_i is properly infinite, each indexing set $T(i)$ is infinite for the sum of a finite number of orthogonal abelian projections is finite. For each positive integer n and for each i the projection P_i may be written as the sum of n orthogonal equivalent projections $F_{i1},\cdots,F_{in}$. If $F_k=\sum_i F_{ik}$ $(1\leqq k\leqq n)$, P is the sum of n equivalent orthogonal projections $F_1, F_2,\cdots,F_n$. Now let Ψ be an irreducible representation of $\mathfrak{A}$ on a Hilbert space H such that the kernel of Ψ is M. The projection P is not in M since $\hat{P}(\zeta)=1$. Because for each positive integer n, P may be written as the sum of n equivalent orthogonal projections H is not finite dimensional. The ideal generated by $[\zeta]$ and I_a is not proper since $M\not\supset I_a$. So there is an A in I_a such that $1-A$ is a member of $[\zeta]$. Thus, $\Psi(A)$ is the identity operator on H. However $\Psi(I_a)$ is the set of all completely continuous operators on the infinite dimensional space H. We have now reached a contradiction.

Conversely, we suppose that P is a central projection such that $M(\mathfrak{A})-h(\mathfrak{A}P)\subset h(I_a)$. There is no loss of generality if we assume $\mathfrak{A}P$ is homogeneous. Indeed, there is a net $\{P_i\}$ of orthogonal central projections such that for each i the algebra $\mathfrak{A}P_i$ is homogeneous. It is sufficient to show that each P_i is properly infinite. We have

$M(\mathfrak{A})-h(\mathfrak{A}P_i)=\{M(\zeta)\in M(\mathfrak{A})\mid \hat{P}_i(\zeta)=1\}\subset\{M(\zeta)\in M(\mathfrak{A})\mid \hat{P}(\zeta)=1\}$ $=M(\mathfrak{A})-h(\mathfrak{A}P)$. So we can assume that $\mathfrak{A}P$ is a homogeneous algebra. There is a net $\{E_i \mid i\in S\}$ of orthogonal equivalent abelian projections of least upper bound P. If S is a finite set, then $P\in I_a$. But in this case every M in $M(\mathfrak{A})-h(\mathfrak{A}P)$ would contain 1 since $1-P\in M$ and since $P\in I_a\subset M$. So the indexing set S in not finite and P is properly infinite.

COROLLARY. *If X is an open set in $h(I_a)$ and if $M\in X$, there is a properly infinite projection P such that $M\in M(\mathfrak{A})-h(\mathfrak{A}P)\subset X$.*

PROOF. Since the spectrum Z of $\mathfrak{Z}$ is a Stonean space, there is an open-closed set Y such that $M\in Y\subset X$. If P is the projection in $\mathfrak{Z}$ such that $Y=\{\zeta\in Z\mid \hat{P}(\zeta)=1\}$ we have $M\in M(\mathfrak{A})-h(\mathfrak{A}P)\subset X$ $\subset h(I_a)$. So P is a properly infinite projection.

Now that we have an explicit representation for the maximal GCR ideal of a Type I AW*-algebra, we can easily show the structure space (i.e. the set of all primitive ideals with the hull-kernel topology) of this ideal is Hausdorff.

Let $\mathfrak{A}$ be a C^*-algebra and I be an (closed two-sided) ideal of $\mathfrak{A}$; for each A in $\mathfrak{A}$ we denote the image of A under the canonical map of $\mathfrak{A}$ onto $\mathfrak{A}/I$ by $A(I)$. The algebra $\mathfrak{A}/I$ is a C^*-algebra under the norm $\|A(I)\|=\text{glb}\{\|A+K\| \mid K\in I\}$. We make use of the following lemma.

LEMMA. *Let $\mathfrak{A}$ be a C^*-algebra; let I and J be ideals of $\mathfrak{A}$; and let $A\in I$. Then $\|A(J)\|=\|A(I\cap J)\|$.*

THEOREM. *The maximal* GCR *ideal of a Type* I AW*-*algebra $\mathfrak{A}$ has a Hausdorff structure space.*

PROOF. Let $P(I_a)=P_a$ be the structure space of I_a taken with the hull-kernel topology. The space P_a is Hausdorff if and only if for each fixed A in I_a the function $f_A=f$ on P_a given by $f(I)=\|A(I)\|$ is continuous. Let $\rho>0$ and A in I_a be given; it is known that the set $\{I\in P_a \mid f(I)\leqq\rho\}$ is closed. It is sufficient to show that the set $\{I\in P_a \mid f(I)<\rho\}$ is open in order to show f is continuous on P_a.

Let J be an ideal in P_a such that $f(J)<\rho$. There is a primitive ideal J_1 in $\mathfrak{A}$ such that $J_1\cap I_a=J$. There is a ζ in the spectrum Z of the center $\mathfrak{Z}$ of $\mathfrak{A}$ such that $J_1\supset[\zeta]$. Thus $J\supset[\zeta]\cap I_a$. There is a representation Ψ of I_a on a Hilbert space H such that (1) the kernel of Ψ is $[\zeta]\cap I_a$ and (2) such that $\Psi(I_a)$ is the set $C(H)$ of completely continuous operators on H. We have that either $\Psi(J)=C(H)$ or that $\Psi(J)=(0)$ because $C(H)$ has no proper ideals. In the first instance $I_a/([\zeta]\cap I_a)=J/([\zeta]\cap I_a)$ or equivalently that $I_a=J$. This is impossible. So we have that $J=I_a\cap[\zeta]$.

For each $B \in \mathfrak{A}$ the function $\zeta' \to \|B([\zeta'])\|$ ($\zeta' \in Z$) is upper semi-continuous. So there is an open and closed set X in Z such that $\zeta \in X \subset \{\zeta' \in Z \mid \|A([\zeta'])\| < \rho\}$ since $\|A([\zeta])\| = f(J) < \rho$. There is a central projection P in $\mathfrak{A}$ such that $X = \{\zeta' \in Z \mid \hat{P}(\zeta') = 1\}$ the set $P_a - h(I_a P)$ is open in P_a. If $I \in P_a - h(I_a P)$ and $I = [\zeta'] \cap I_a$ for some $\zeta' \in Z$, we have $[\zeta'] \cap I_a \not\supset I_a P$. Since $[\zeta'] \cap I_a \supset [\zeta'] \cdot I_a$ we have $\hat{P}(\zeta') = 1$. Thus, if $I \in P_a - h(I_a P)$, we have $\|A(I)\| = \|A(I_a \cap [\zeta'])\| = \|A([\zeta'])\| < \rho$. Also we have that $\hat{P}(\zeta) = 1$. If $J \supset I_a P$ then for each B in I_a $\|B(J)\| = \|B(I_a \cap [\zeta])\| = \|BP([\zeta])\| = 0$. Thus, $J = I_a$. We conclude that $J \not\supset I_a P$ so $J \in P_a - h(I_a P)$. This completes the proof.

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