PERIODIC SOLUTIONS OF FOURTH-ORDER DIFFERENTIAL EQUATIONS

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The object of this paper is to prove the following result.

THEOREM. *Let* $f: R^2 \times R^2 \to R^2$ *and* $g: R^2 \to R^2$ *be* C^1 *functions and consider the differential system*

$$(1) \qquad \begin{aligned} x' &= f(x, y), \\ y' &= g(y), \end{aligned}$$

where $x, y \in R^2$, $x' = dx/dt$ *and* $y' = dy/dt$. *If there is a positively compact solution* $(x(t), y(t))$ *of* (1), *then* (1) *has a periodic solution.*

Recall that a solution $(x(t), y(t))$ is said to be positively compact if $(x(t), y(t))$ remains in a compact set for all $t \geqq 0$, and the solution is compact, if it remains in a compact set for all t in R.

PROOF. First we note that we can assume that $\|f(x, y)\| \leqq 1$, for all x and y, where $\|\cdot\|$ denotes a norm on R^2. Indeed, if this were not true we could replace (1) with

$$(2) \qquad \begin{aligned} x' &= (1 + \|f\|)^{-1} f(x, y) = \hat{f}(x, y), \\ y' &= g(y). \end{aligned}$$

Then the solution curves of (1) agree with those of (2) and $\|\hat{f}\| \leqq 1$.

Now let $(x(t), y(t))$ be a positively compact solution of (1). Then $y(t)$ is a positively compact solution of $y' = g(y)$ on R^2, and by the Poincaré-Bendixson Theory, cf. [2, pp. 394–395], the positive limit set L_y^+ of $y(t)$ in R^2 is nonempty and contains a periodic solution $\hat{y}(t)$ of $y' = g(y)$. (The solution $\hat{y}$ may be an equilibrium point of $y' = g(y)$.) Now consider the second-order, periodic, differential equation

$$(3) \qquad x' = f(x, \hat{y}(t)).$$

We claim that (3) has a compact solution. Indeed, since $(x(t), y(t))$ is a positively compact solution of (1), its positive limit set $\Omega(x, y)$ in R^4 is nonempty, compact and invariant, cf. [4, pp. 338–340]. Let $P: R^2 \times R^2 \to R^2$ be the mapping defined by $P: (x, y) \to y$. Then it is clear that $P(\Omega(x, y)) = L_y^+$. With $\hat{y}(t)$ given above, choose $\tilde{x}(t)$ so that $(\tilde{x}, \hat{y}) \in \Omega(x, y)$. Then $\tilde{x}(t)$ is a compact solution of (3).

Since $\|f\| \leqq 1$, the solutions of (3) can be continued for all time

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$t \geqq 0$. It then follows from Massera's Theorem [3, Theorem 2] that (3) has a periodic solution $\hat{x}(t)$, and further, the periods of $\hat{x}(t)$ and $\hat{y}(t)$ agree. Consequently, $(\hat{x}(t), \hat{y}(t))$ is a periodic solution of (1). This completes the proof of the theorem.

REMARKS 1. It is not necessary to assume that f and g are C^1 functions. What is needed is that f and g be continuous and that the solutions of (1) be unique.

2. The domain for the y-variable can be changed to an open subset W of R^2. That is, f and g are defined on $R^2 \times W$ and W, respectively. Since Massera's Theorem makes use of a fixed point theorem of Brouwer, cf. [1], for mappings of R^2 into R^2, it does not appear that the domain of the x-variable can be similarly changed.

3. Finally, it should be noted that the above theorem admits the following generalization. Consider the equation

$$z' = F(z), \tag{4}$$

on R^4, where $F: R^4 \to R^4$ is of class C^1. Assume that there is a C^1 diffeomorphism $\phi: R^4 \to R^4$ that changes (4) into (1). Since the solution curves of (4) are mapped onto those of (1) by ϕ, we can then conclude that if (4) has a positively bounded solution, it has a periodic solution.

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