REGULAR TOPOLOGIES AND POSETS

W. J. PERVIN AND H. J. BIESTERFELDT, JR.[1]

Let $(x_m: m \in D)$ be a net, which we will call the *iterate net*, in a topological space X such that, for each $m \in D$, there is a net $(x_d^m: d \in D_m) \to x_m$. We will call the net

$$\left(x^m_{\beta(m)}: \langle m, \beta \rangle \in D \times \prod_{m \in D} D_m \right),$$

where the product set is directed by the product order, the *composite net* of the system of nets. It is well known (see [5, p. 69]) that if the iterate net converges then the composite net converges to the same limit. Indeed, this property helps characterize topologies through convergence classes (see [5, p. 74]). We will show that the converse of this iterated limit theorem characterizes regular spaces.

THEOREM A. *A topological space is regular iff any iterate net converges to the limit of the composite net whenever that limit exists.*

PROOF. Let X be a regular topological space and suppose the composite net of a system of nets, with the above notation, converges to a point $x \in X$. Let G be any open set containing x. By regularity, we may choose an open set G^* such that $x \in G^* \subset c(G^*) \subset G$. Now there exists an element $\langle m^*, \beta^* \rangle$ in the product directed set such that $x^m_{\beta(m)} \in G^*$ for all $\langle m, \beta \rangle \geqq \langle m^*, \beta^* \rangle$. Since we have $(x_d^m: d \in D_m) \to x_m$ and each $x^m_{\beta(m)} \in G^*$ for $m \geqq m^*$, it follows that $x_m \in c(G^*) \subset G$ for $m \geqq m^*$ and the limit of the iterate net is x. Conversely, suppose X is not regular. Then there exists a point x and an open set G^* containing it such that $c(G) \not\subset G^*$ for any open set G containing x. Let $\{G_m: m \in D\}$ be the family of all open sets containing x, directed by inclusion. Since $c(G_m) \not\subset G^*$ for each $m \in D$, there exists a point $x_m \in c(G_m) - G^*$. Since $x_m \in c(G_m)$, there exists a net $(x_d^m: d \in D_m)$ in G_m converging to x_m. Since $x_m \notin G^*$ for each $m \in D$, the iterate net $(x_m: m \in D)$ cannot converge to x. The composite net does converge to x, however, as the following shows. Let G be an arbitrary open set containing x. Then $G = G_d$ for some $d \in D$. Let $\langle d, \alpha \rangle$ be a member of the product directed set with α fixed but arbitrary. Then if $\langle m, \beta \rangle \geqq \langle d, \alpha \rangle$, then $x^m_{\beta(m)} \in G_m \subset G_d = G$. q.e.d.

Received by the editors December 19, 1965.

[1] This work was partially supported by the National Science Foundation under NSF grants GP 4432 and GY 469.

This characterization of regularity was motivated by results of Dieudonné [2] and of Cook and Fischer [3] who use a product of filters and a "compression" operator on families of filters, respectively.

Since the property of the theorem depends only on a notion of convergence for nets, we may call a poset *regular* with respect to its order convergence (or *o*-convergence as defined in [1, p. 60]) iff any iterate net converges to the limit of the composite net whenever that limit exists. We recall that a point x in a poset is a limit of a net $(x_m: m\in D)$ with respect to its order convergence iff lim sup x_m $=$lim inf $x_m=x$, where these limits are taken in the completion by cuts.

THEOREM B. *Every poset is regular in its order convergence.*

PROOF. In the above notation, we need only prove that lim sup x_m $\leqq$lim sup $x^m_{\beta(m)}$ since then, by duality,

$$x = \liminf x^m_{\beta(m)} \leqq \liminf x_m \leqq \limsup x_m \leqq \limsup x^m_{\beta(m)} = x$$

and so $(x_m: m\in D)\to x$ as desired. We must show, then, that

$$\bigwedge_{m^*} \bigvee_{m\geqq m^*} x_m \leqq \bigwedge_{\langle m^*,\beta^*\rangle} \bigvee_{\langle m,\beta\rangle\geqq\langle m^*,\beta^*\rangle} x^m_{\beta(m)}.$$

Let us fix $\langle m^*, \beta^*\rangle$, which gives us an element

$$\bigvee_{\langle m,\beta\rangle\geqq\langle m^*,\beta^*\rangle} x^m_{\beta(m)}$$

over which we take the infimum on the right-hand side, and we shall show that (for the same fixed m^*), the element

$$\bigvee_{m\geqq m^*} x_m$$

is smaller, and so the infima are comparable as designated. We calculate that

$$\begin{aligned}\bigvee_{m\geqq m^*} x_m = \bigvee_{m\geqq m^*} \bigwedge_{d^*} \bigvee_{d\geqq d^*} x^m_d &\leqq \bigvee_{m\geqq m^*} \bigwedge_{d^*\geqq\beta^*(m)} \bigvee_{d\geqq d^*} x^m_d \\ \leqq \bigvee_{m\geqq m^*} \bigvee_{d\geqq\beta^*(m)} x^m_d &\leqq \bigvee_{\langle m,\beta\rangle\geqq\langle m^*,\beta^*\rangle} x^m_{\beta(m)},\end{aligned}$$

where the final inequality follows from the fact that for any element x^m_d on the left, we have $m\geqq m^*$ and $d\geqq\beta^*(m)$. Then by defining $\beta(k)=d$ if $k=m$ and $=\beta^*(m)$ otherwise, we have $x^m_{\beta(m)}=x^m_d$ with $\langle m, \beta\rangle\geqq\langle m^*, \beta^*\rangle$; thus the element appears on the right. q.e.d.

An application of this result is an immediate proof of a result of DeMarr [**4**]:

COROLLARY. *Every O-space is a regular Hausdorff space.*

PROOF. The Hausdorff condition is immediate since limits are unique in a complete lattice. Suppose we have a system of nets with the composite net converging with respect to the topology. Since the space is an O-space, by definition it is homeomorphic to a subset of a complete lattice with each net converging with respect to the topology iff it o-converges to that limit. Since the complete lattice is regular by Theorem B, the iterate net o-converges to the same limit and hence also with respect to the topology. By Theorem A, the space is regular.

BIBLIOGRAPHY

1. G. Birkhoff, *Lattice theory*, rev. ed., Amer. Math. Soc. Colloq. Publ. Vol. 25, Amer. Math. Soc., Providence, R. I., 1948.

2. N. Bourbaki and J. Dieudonné, *Note de tératopologie.* II, Rev. Sci. **77** (1939), 180–181.

3. C. H. Cook and H. R. Fischer, *Regular convergence spaces*, Symposium on Convergence Structures, Preprint No. 20, University of Oklahoma, Norman, 1965.

4. R. E. DeMarr, *Order convergence and topological convergence*, Proc. Amer. Math. Soc. **16** (1965), 588–590.

5. J. L. Kelley, *General topology*, Van Nostrand, New York, 1955.

UNIVERSITY OF WISCONSIN-MILWAUKEE