LINEAR TRANSFORMATIONS THAT PRESERVE THE PERMANENT

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1. **Introduction.** Let $M_m(F)$ be the vector space of m square matrices over a field F. If X belongs to $M_m(F)$ we denote by x_{ij} the element in the ith row and jth column of X. Let S_m be the symmetric group of degree m. The permanent of a matrix X in $M_m(F)$, denoted by per X, is defined by

$$\text{per } X = \sum_{\sigma \in S_m} \prod_{i=1}^{m} x_{i\sigma(i)}.$$

The permanent is of interest in combinatorial problems [4] and has recently been the subject of several investigations [1], [3]. In a recent paper [1] Marcus and May proved the following:

THEOREM. *Let T be a linear transformation on $M_m(F)$ into itself and suppose* per $T(X) =$ per X *for all X in $M_m(F)$. Then, if $m > 2$, there exist permutation matrices P and Q and diagonal matrices D and L such that* per $DL = 1$ *and either* $T(X) = DPXQL$ *or* $T(X) = DPX'QL$ *(here X' denotes the transpose of the matrix X).*

In their proof of this result Marcus and May used the rth permanental compound, a $(C_{m,r})$ square matrix analogous to the well-known rth determinantal compound. The purpose of this note is to prove the above theorem in a somewhat more direct way, avoiding the use of compound matrices. The author would like to thank Professor Marvin Marcus for suggesting this problem to him.

2. **Proof of Theorem.** In all that follows T is a fixed linear transformation on $M_m(F)$ to itself that satisfies per $T(X) =$ per X for all X. We first establish some useful lemmas, some of which may be found in Marcus and May [1].

LEMMA 1 [1, LEMMA 2]. *T is nonsingular.*

Suppose X is in $M_m(F)$ and $1 \le i_1 < \cdots < i_t \le m$, $1 \le j_1 < \cdots < j_t \le m$, then by $X[i_1, \cdots, i_t | j_1, \cdots, j_t]$ we mean the t square matrix Y defined by $Y_{rs} = X_{i_r j_s}$ and by R_i (R^i) we mean those matrices with row i (column j) zero.

LEMMA 2. *Let A belong to $M_m(F)$ and suppose z is an indeterminate*

over F. If for all X in $M_m(F)$, deg per $(X+zA)\leqq 1$ then for all $i<j$, $k<l$ per $A\,[i,j\,|\,k,\,l]=0$.

PROOF. A straightforward computation shows that the coefficient of z^2 in the polynomial per $(X+zA)$ is

$$\sum_{\sigma\in S_m}\left(\sum_{i<j}a_{i\sigma(i)}a_{j\sigma(j)}\prod_{t=1;\,t\neq i,j}^{m}x_{t\sigma(t)}\right).$$

For fixed $i<j$ let σ in S_m be the permutation interchanging i and j. Choose B in $M_m(F)$ such that $b_{tt}=1$ for $t\neq i,\,j$ and $b_{rs}=0$ otherwise. Then the coefficient of z^2 in per $(B+zA)$ is per $A\,[ij\,|\,ij]$ which by assumption is zero. The result now follows if we observe that for any two permutation matrices P and Q, per $PAQ=$ per A.

A result similar to the above has been used [2] in considering a similar problem for the determinant function.

LEMMA 3 [1]. *Suppose A is in $M_m(F)$. Then* per $A\,[i,j\,|\,k,\,l]=0$ *for all $i<j$, $k<l$ if and only if either*

(1) *A belongs to $R_i(R^i)$ for some i.*

(2) *There exist $r<s$, $u<v$ such that*

$$A\,[u,\,v\,|\,r,\,s]=\begin{pmatrix} x & -zx/y \\ y & z \end{pmatrix}$$

for some $x,\,y,\,z\neq 0$ in F and all other entries in A are zero.

SKETCH OF PROOF. Clearly if A is of either form then per $A\,[i,j\,|\,k,\,l]=0$ for all $i<j$, $k<l$. Suppose A is not of either form and all 2 square subpermanents of A vanish. Then there exist $i<j$, $k<l$ such that $a_{ik}\neq 0$, $a_{jl}\neq 0$ and $(\alpha,\,\beta)$ different from any of $(i,\,k)$, $(i,\,l)$, $(j,\,k)$, $(j,\,l)$ such that $a_{\alpha\beta}\neq 0$. If we consider the submatrices $A\,[i,j\,|\,k,\,l]$, $A\,[\alpha,\,i\,|\,\beta,\,l]$ and $A\,[\alpha,\,j\,|\,\beta,\,k]$ and use the fact that the permanents of these must vanish we soon arrive at a contradiction.

We now turn to the proof of the theorem. Let z be an indeterminate over F. If $A\in R_i\,(R^i)$ then, by Lemma 2, deg per $(X+zA)\leqq 1$ for all X. Therefore deg per $(T(X)+zT(A))\leqq 1$ for all X. By Lemma 1, T is nonsingular hence as X runs over $M_m(F)$ so does $T(X)$. It now follows that $T(A)$ is of one of the forms described in Lemma 3. We show that the second form cannot occur. To this end it is convenient to consider the entries of $T(A)$ as linear forms in the entries of A. Let $z_1,\,\cdots,\,z_m$ be indeterminates over F. We let E_{ij} be the matrix with a 1 in the (i,j) position and zeros elsewhere

$$Z=\sum_{t=1}^{m}z_tE_{it};$$

$$(T(Z))_{rs} = L_{rs}(z_1, \cdots, z_m).$$

It follows that $T(Z)$ must have one of the forms described in Lemma 3 for if not, we could choose $z_t = a_t \in F$ such that

$$A = \sum_{t=1}^{m} a_t E_{it} \in R_i$$

and $T(A)$ would not have the required form. Suppose $T(Z)$ has the second form. Then for some $i < j$, $k < l$ we have

$$T(Z)[i, j \mid k, l] = \begin{pmatrix} L_{ik}(z_1, \cdots, z_m) & L_{il}(z_1, \cdots, z_m) \\ L_{jk}(z_1, \cdots, z_m) & L_{jl}(z_1, \cdots, z_m) \end{pmatrix}$$

and all other elements of $T(Z)$ are zero. Further, since deg per $(Y + zT(Z)) \leq 1$ for all Y (as a function of z), all 2 square subpermanents of $T(Z)$ must vanish. This observation, in conjunction with the unique factorization theorem for polynomials in several variables, shows that either

$$L_{ik}(z_1, \cdots, z_m) = L_{il}(z_1, \cdots, z_m)$$

and

$$L_{jk}(z_1, \cdots, z_m) = - L_{jl}(z_1, \cdots, z_m)$$

or

$$L_{ik}(z_1, \cdots, z_m) = - L_{jk}(z_1, \cdots, z_m)$$

and

$$L_{il}(z_1, \cdots, z_m) = L_{jl}(z_1, \cdots, z_m)$$

to within a permutation of rows and columns.

The two linear forms L_{ik} and L_{jl} must involve all of the indeterminates because T is nonsingular. We have assumed $m \geq 3$ so we can choose $a_1, \cdots, a_m$ in F not all zero such that $L_{ik}(a_1, \cdots, a_m) = L_{jl}(a_1, \cdots, a_m) = 0$. This however implies that T is singular, a contradiction. It now follows that $T(R_i) = R_j$ or R^j for some j and that $T(R^t) = R_k$ or R^k for some k. If we recall that T is nonsingular and onto then it is not hard to check that either

(i) $T(R_i) = R^{i_i}$ and $T(R^t) = R_{k_t}$ or

(ii) $T(R_i) = R_{j_i}$ and $T(R^t) = R^{k_t}$

for all i, t and that $j_i = \sigma(i)$, $k_t = \tau(t)$ for suitable permutations $\sigma, \tau \in S_m$. Suppose the first case occurs. Then

$$T(E_{ij}) = \sum_{s=1}^{m} c_s E_{s\sigma(i)} = \sum_{r=1}^{m} d_r E_{\tau(j)r}.$$

Now $(s, \sigma(i)) = (\tau(j), r)$ if and only if $s = \tau(j)$, $\sigma(i) = r$. The matrices E_{kl}, $k, l = 1, \cdots, m$ are linearly independent so it follows that $T(E_{ij})$ $b_{ij}E_{\tau(j)\sigma(i)}$ for some $b_{ij} \in F$.

Let P, Q and B be defined by

$$P_{ij} = \delta_{i\sigma^{-1}(j)}, \qquad Q_{ij} = \delta_{i\tau(j)}, \qquad B_{ij} = b_{ij}.$$

Then a straightforward computation shows that

$$T(X) = B * (PX'Q)$$

where $*$ is the Hadamard product defined by $(R*S)_{ij} = R_{ij}S_{ij}$, and $\delta_{rs} = 1$ if $r = s$ and 0 otherwise.

If the second case occurs we may show in a similar way that $T(X) = B * (PXQ)$ for permutation matrices P and Q and a suitable matrix B.

By choosing X to be an appropriate permutation matrix we find that for each $\omega \in S_m$

$$\prod_{i=1}^{m} b_{i\omega(i)} = 1;$$

so for any i, j, $b_{ij} \neq 0$. Hence by choosing ω and μ to be appropriate transpositions we have $b_{ir}b_{js} = b_{is}b_{jr}$ for all i, j, r, s. Set $a_i = b_{i1}/b_{11}$; $c_j = b_{1j}$ then $a_ic_j = b_{i1}b_{1j}/b_{11} = b_{ij}$. Hence if $A = \text{diag}(a_1, \cdots, a_m)$; $C = \text{diag}(c_1, \cdots, c_m)$ we have that for any $X \in M_m(F)$

$$B * X = AXC.$$

Also per $AC = \prod_{i=1}^{m} a_ic_i = \prod_{i=1}^{m} b_{ii} = 1$.

This completes the proof of the Theorem.

REFERENCES

1. M. Marcus and F. May, *The permanent function*, Canad. J. Math. 14 (1962), 177–189.

2. M. Marcus, *Linear operations on matrices*, Amer. Math. Monthly 69 (1962), 837–847.

3. M. Marcus and H. Minc, *On the relation between the permanent and the determinant*, Illinois J. Math. 5 (1961), 376–381.

4. H. J. Ryser, *Compound and induced matrices in combinatorial analysis*, Proc. Sympos Appl. Math., Vol. 10, pp. 149–167, Amer. Math. Soc., Providence, R. I., 1960.

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