ON THE RELATIONSHIP BETWEEN ZERO ENTROPY AND QUASI-DISCRETE SPECTRUM FOR AFFINE TRANSFORMATIONS

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Introduction. If T is a measure-preserving transformation of a Lebesgue space $(M, \mathfrak{B}, m)$ let $\pi(T)$ be the maximum partition such that $T_{\pi(T)}$ has zero entropy. In §2 we prove a result, which was stated without proof in [12], concerning the behaviour of the partitions $\pi(T_{\xi_n})$ associated with an increasing sequence of invariant measurable partitions $\{\xi_n\}$.

If T is an ergodic affine transformation of a compact connected metric abelian group G let $\eta(T)$ be the maximum partition such that $T_{\eta(T)}$ has quasi-discrete spectrum. In §3 we prove, using the result of §2 and a method introduced by Rohlin in [8], that $\eta(T) = \pi(T)$. This result was first obtained by Parry [6] who also proved that the maximum partition $\gamma(T)$ such that $T_{\gamma(T)}$ has the distal property is also $\pi(T)$.

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1. Preliminaries.

1.1. *The partition $\pi(T)$.* For the definitions, notations and theory of Lebesgue spaces and entropy cf. [9], [10], [11].

Let $\mathcal{Z}$ be the set of all measurable partitions of the Lebesgue space $(M, \mathfrak{B}, m)$ which have finite entropy and let T be a (possibly many to one) measure-preserving transformation of $(M, \mathfrak{B}, m)$. The maximum partition $\pi(T)$ such that $h(T_{\pi(T)}) = 0$ is given by $\pi(T) = \bigvee_{\xi \in \mathcal{Z}} (\bigwedge_{n=1}^{\infty} T^{-n}\xi^-)$ where $\xi^- = \bigvee_{k=1}^{\infty} T^{-k}\xi$ [11].

We have $T^{-1}\pi(T) = \pi(T)$, $\pi(T^n) = \pi(T)$ $n > 0$, and if T is one to one $\pi(T^{-1}) = \pi(T)$. If $\xi \in \mathcal{Z}$ and $\xi \leq \pi(T)$ then $\xi \leq \xi^-$.

T is said to have completely positive entropy if $\pi(T) = \nu$, the trivial partition. If T and T' are measure-preserving transformations of the Lebesgue spaces $(M, \mathfrak{B}, m)$ and $(M', \mathfrak{B}', m')$ respectively and if there exists a measure-preserving transformation ϕ of $(M, \mathfrak{B}, m)$ onto $(M', \mathfrak{B}', m')$ such that $\phi T = T' \phi$, then T' is said to be a factor transformation of T and $h(T') \leq h(T)$ [11].

1.2. *Affine transformations and group partitions.* Let G be a compact connected metric abelian group equipped with completed Haar measure. G is a Lebesgue space [10]. All group operations are written

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additively. Since G is connected its character group Γ has no elements of finite order [7]. Endomorphisms of G and their duals will be denoted by the same symbol.

An affine transformation of G is a transformation of the form $T(g) = a + A(g)$, $g \in G$, where $a \in G$ and A is a continuous endomorphism of G onto G . Affine transformations preserve Haar measure. $T(g) = a + A(g)$, $g \in G$, is ergodic if and only if the smallest closed subgroup containing a and $(A - I)G$ is G and the only finite orbits under A in the character group are fixed elements [3]. Since G is connected, if T is ergodic T is totally ergodic (i.e., T^n is ergodic for all $n > 0$).

If $BG = G$, where $B = A - I$, then $T(b + g) = b + A(g)$, $g \in G$, where $B(b) = -a$ and T has completely positive entropy. (This follows from the fact that A has completely positive entropy [8].)

Let H be a subgroup of G . (All subgroups are assumed closed.) The measurable partition of G into cosets of H will be denoted by $\zeta(H)$ [8]. Such a partition is called a group partition. If $H_1 \subset H_2$ then $\zeta(H_1) \geq \zeta(H_2)$, $\zeta(G) = \nu$, and $\zeta(0) = \epsilon$, the partition of G into points. If $H_1 \subset H_2 \subset \dots$ then $\zeta(\bigvee H_n) = \bigwedge \zeta(H_n)$ where $\bigvee H_n$ denotes the smallest closed subgroup containing all the H_n , and if $H_1 \supset H_2 \supset \dots$ then $\zeta(\bigwedge H_n) = \bigvee \zeta(H_n)$ where $\bigwedge H_n$ is the intersection of the groups H_n .

The partition $\alpha(T) = \bigwedge_{n=0}^{\infty} T^{-n}\epsilon$ is the finest partition of G which satisfies $T^{-1}\alpha(T) = \alpha(T)$. In particular $\pi(T) \leq \alpha(T)$. $\alpha(T)$ is a group partition since

$$\alpha(T) = \alpha(A) = \bigwedge_{n=0}^{\infty} A^{-n}\epsilon = \zeta\left(\bigvee_{n=1}^{\infty} \text{kernel } A^n\right).$$

1.3. Affine transformations with quasi-discrete spectrum. The definition of a totally ergodic measure-preserving transformation with quasi-discrete spectrum is given in [1] and the definition of a homeomorphism with quasi-discrete spectrum is in [2]. Let $T(g) = a + A(g)$ be an ergodic affine transformation of a compact connected metric abelian group G with character group Γ . If $\Gamma'_n = \{\gamma \in \Gamma \mid B^n\gamma = 0\}$ where $B = A - I$, then the group of quasi-eigenfunctions of order n is $K \times \Gamma'_n$ where K is the circle group. T has quasi-discrete spectrum if and only if $\bigcup_{n=1}^{\infty} \Gamma'_n = \Gamma$ [4].

The maximum partition $\eta(T)$ such that $T_{\eta(T)}$ has quasi-discrete spectrum is the partition of G into cosets of $\text{ann}(\bigcup_{n=1}^{\infty} \Gamma'_n)$ where $\text{ann}(\bigcup_{n=1}^{\infty} \Gamma'_n)$ is the annihilator of $\bigcup_{n=1}^{\infty} \Gamma'_n$ i.e. $\eta(T) = \zeta(\text{ann}(\bigcup_{n=1}^{\infty} \Gamma'_n))$ [4].

Abramov [1] has proved that every totally ergodic measure-preserving transformation of a Lebesgue space with quasi-discrete

spectrum has zero entropy. Therefore for $T(g) = a + A(g)$, $g \in G$, we have $\eta(T) \leq \pi(T)$.

2. A limit theorem for $\pi(T)$. If ζ is a measurable partition of the Lebesgue space $(M, \mathcal{B}, m)$ let H_ζ denote the projection of M onto M/ζ , i.e. H_ζ maps a point of M onto the element of ζ to which it belongs. If ζ is a measurable partition of M such that $T^{-1}\zeta \leq \zeta$ then let $\pi'(T_\zeta)$ denote $H_\zeta^{-1}\pi(T_\zeta)$.

THEOREM 1. *Let T be a measure-preserving transformation (possibly many to one) of a Lebesgue space $(M, \mathcal{B}, m)$ and let $\{\zeta_n\}$ be a sequence of measurable partitions of M such that $\zeta_1 \leq \zeta_2 \leq \dots$, $\bigvee_n \zeta_n = \epsilon$, and $T^{-1}\zeta_n \leq \zeta_n$ for all n . Then $\pi'(T_{\zeta_1}) \leq \pi'(T_{\zeta_2}) \leq \dots$ and $\bigvee_n \pi'(T_{\zeta_n}) = \pi(T)$.*

PROOF. Since $\pi'(T_{\zeta_n}) = \pi(T) \wedge \zeta_n$ we have $\pi'(T_{\zeta_1}) \leq \pi'(T_{\zeta_2}) \leq \dots$. Let $\pi_1(T) = \bigvee_n \pi'(T_{\zeta_n})$.

Since $\pi'(T_{\zeta_n}) \leq \pi(T)$ for all n , we have $\pi_1(T) \leq \pi(T)$. It remains to show that $\pi_1(T) \geq \pi(T)$. Let $\xi \in \mathcal{Z}$, $\xi \leq \pi(T)$ and $\eta \in \mathcal{Z}$. Then since $\xi \vee \xi^- = \xi^-$ we have

$$\begin{aligned} H(\eta/\eta^-) &\geq H(\eta/\eta^- \vee \xi) \geq H(\eta/\eta^- \vee \xi \vee \xi^-) = H(\eta/\eta^- \vee \xi^-) \\ &= H((\eta \vee \xi)/(\eta^- \vee \xi^-)) = h(T, \eta \vee \xi) \geq h(T, \eta) = H(\eta/\eta^-), \end{aligned}$$

i.e.

$$H(\eta/\eta^- \vee \xi) = H(\eta/\eta^-).$$

But

$$H(\xi/\eta^-) + H(\eta/\xi \vee \eta^-) = H(\xi \vee \eta/\eta^-) = H(\eta/\eta^-) + H(\xi/\eta \vee \eta^-)$$

and therefore

$$H(\xi/\eta^-) = H(\xi/\eta \vee \eta^-).$$

Consequently $H(\xi/T^{-k}\eta^-) = H(\xi/\eta \vee \eta^-)$ for all $k > 0$, and if $\eta_\infty^- = \bigwedge_{k=1}^\infty T^{-k}\eta^-$ we have [9]

$$H(\xi/\eta_\infty^-) = \lim_{k \rightarrow \infty} H(\xi/T^{-k}\eta^-) = H(\xi/\eta \vee \eta^-).$$

Let $\xi \in \mathcal{Z}$, $\xi \leq \pi(T)$ be fixed. Given $\delta > 0$, $\exists \eta \in \mathcal{Z}$, $\eta \leq \zeta_n$ for some n , such that $H(\xi/\eta) < \delta$.

Since $\eta_\infty^- = \bigwedge_{k=1}^\infty T^{-k}\eta^- \leq \pi'(T_{\zeta_n}) \leq \pi_1(T)$, we have $H(\xi/\pi_1(T)) \leq H(\xi/\eta_\infty^-) = H(\xi/\eta \vee \eta^-) \leq H(\xi/\eta) < \delta$. Therefore $H(\xi/\pi_1(T)) = 0$ and $\xi \leq \pi_1(T)$. Since this is true for all $\xi \in \mathcal{Z}$, $\xi \leq \pi(T)$, we have $\pi(T) \leq \pi_1(T)$.

COROLLARY. *If T is a measure-preserving transformation of a Lebesgue space $(M, \mathcal{B}, m)$ and if $\{\zeta_n\}$ is a sequence of measurable parti-*

tions of M such that $\zeta_1 \leq \zeta_2 \leq \dots$, $\bigvee_n \zeta_n = \zeta$, and $T^{-1}\zeta_n \leq \zeta_n$ for all n , then $\bigvee_n \pi'(T_{\zeta_n}) = \pi'(T_{\zeta})$.

PROOF. Apply Theorem 1 to the transformation T_{ζ} on M/ζ .

3. Zero entropy and quasi-discrete spectrum for affine transformations.

THEOREM 2. *Let G be a compact connected metric abelian group and let $T(g) = a + A(g)$, $g \in G$, be an ergodic affine transformation of G . The maximum partition $\pi(T)$ such that $T_{\pi(T)}$ has zero entropy and the maximum partition $\eta(T)$ such that $T_{\eta(T)}$ has quasi-discrete spectrum coincide.*

PROOF. We first show that $\pi(T) = \eta(T)$ if, and only if, $\pi(T_{\alpha(T)}) = \eta(T_{\alpha(T)})$. From §1.2 $\alpha(T) = \zeta(F)$ for some subgroup F of G satisfying $AF = F$. Therefore $T_{\alpha(T)}$ is the one to one ergodic affine transformation $T_{\alpha(T)}(g') = a' + A'(g')$, $g' \in G/F$, where A' is the automorphism induced by A in G/F and a' is the coset of F containing a . If $H_{\alpha(T)}$ is the projection map of G onto G/F then $H_{\alpha(T)}^{-1}\pi(T_{\alpha(T)}) = \alpha(T) \wedge \pi(T)$ and $H_{\alpha(T)}^{-1}\eta(T_{\alpha(T)}) = \alpha(T) \wedge \eta(T)$. Since $\eta(T) \leq \pi(T) \leq \alpha(T)$ we conclude that $\pi(T) = \eta(T)$ if and only if $\pi(T_{\alpha(T)}) = \eta(T_{\alpha(T)})$.

We divide the proof of the theorem into three parts.

(i) Suppose that G is a finite-dimensional torus. We can suppose, without loss of generality, that T and A are one to one. This follows because $T_{\alpha(T)}$ is a one to one ergodic affine transformation of the finite-dimensional torus G/F and $\pi(T) = \eta(T)$ if and only if $\pi(T_{\alpha(T)}) = \eta(T_{\alpha(T)})$.

The set of periodic points of an automorphism of a finite-dimensional torus is dense i.e. if $A_n = \{g \in G \mid A^n g = g\}$, $n > 0$, then $\bigcup_{n=1}^{\infty} A_n$ is dense in G [8]. If $\phi(g) = g_1 + g$ where $g_1 \in A_n$, then $\phi T^n = T^n \phi$ and hence $\phi^{-1}\pi(T^n) = \pi(T^n)$. Since $\pi(T^n) = \pi(T)$, $n > 0$, $\pi(T)$ is completely invariant under a dense set of translations and is therefore a group partition $\pi(T) = \zeta(N)$ [8].

$G_1 = G/N$ is a finite-dimensional torus with character group $\Gamma_1 = \text{ann}(N)$, and $T_{\pi(T)}$ is the ergodic affine transformation $T_1(g) = a_1 + A_1(g)$, $g \in G_1$, where A_1 is the automorphism of G_1 induced by A and a_1 is the coset of N containing a .

$\{B_1^m G_1, m \geq 0\}$, where $B_1 = A_1 - I$, is a decreasing sequence of subgroups of G_1 and therefore $\{\text{ann}(B_1^m G_1), m \geq 0\}$ is an increasing sequence of subgroups of Γ_1 . Since Γ_1 is a finitely generated free abelian group $\text{ann}(B_1^n G_1) = \text{ann}(B_1^{n+1} G_1)$ for some n and $B_1^n G_1 = B_1^{n+1} G_1$.

Define an affine transformation S on the compact connected metric abelian group $B_1^n G_1$ by $S(B_1^n g) = B_1^n a + A_1(B_1^n g)$, $g \in G_1$. Since A_1 and B_1 commute we have $SB_1^n = B_1^n T_1$. Since $B_1(B_1^n G_1) = B_1^{n+1} G_1$, S has com-

pletely positive entropy (see §1.2). But $h(T_1) = h(T_{\pi(T)}) = 0$ and therefore S , being a factor transformation of T_1 , has zero entropy (see §1.1).

Therefore $B_1^n G_1 = \{0\}$, and

$$\Gamma'_n = \text{ann}(B_1^n G_1) = \{\gamma \in \Gamma_1 \mid \gamma(B_1^n g) = 0, \text{ all } g \in G_1\} = \Gamma_1,$$

i.e. every element of Γ_1 is a quasi-eigenfunction of order n (see §1.3), and therefore T_1 has quasi-discrete spectrum. Therefore $\eta(T) \geq \pi(T)$ and hence $\eta(T) = \pi(T)$.

(ii) Suppose now that Γ , the character group of G , is finitely generated with respect to the endomorphism A , i.e. there exists a finite number of elements $\gamma_1, \dots, \gamma_r$ of Γ such that every element of Γ is of the form $p_1(A)\gamma_1 + \dots + p_r(A)\gamma_r$ where $p_i \in \mathbb{Z}(A)$, the ring of polynomials in A with integer coefficients.

We can again suppose, without loss of generality, that T and A are one to one. This follows since Γ is a finitely generated module over the Noetherian ring $\mathbb{Z}(A)$ and is therefore a Noetherian module [13]. Since $\alpha(T) = \zeta(F)$, $\text{ann}(F)$ is a submodule of Γ and hence is finitely generated over $\mathbb{Z}(A)$, i.e. there exists $\beta_1, \dots, \beta_s \in \Gamma$ such that

$$\text{ann}(F) = \{q_1(A)\beta_1 + \dots + q_s(A)\beta_s \mid q_i \in \mathbb{Z}(A), i = 1, \dots, s\}.$$

Since $A(\text{ann}(F)) = \text{ann}(F)$, $\text{ann}(F)$, the character group of G/F , is finitely generated with respect to $A_{\alpha(T)}$. Since $A_{\alpha(T)}$ and $T_{\alpha(T)}$ are one to one and $\pi(T) = \eta(T)$ if and only if $\pi(T_{\alpha(T)}) = \eta(T_{\alpha(T)})$, the assertion follows.

Let

$$Y_m = \{p_1(A)\gamma_1 + \dots + p_r(A)\gamma_r \mid p_i \in \mathbb{Z}(A) \text{ and degree of } p_i \leq m, i = 1, \dots, r\}$$

Y_m is a subgroup of Γ and $\bigcup_{m=0}^{\infty} Y_m = \Gamma$. Choose m such that $A^{-1}\gamma_i \in Y_m$, $i = 1, \dots, r$. Then $A^{-1}Y_m \subset Y_m$ and if $H = \text{ann}(Y_m)$, $A^{-1}H \subset H$. Since $A^n Y_m$, $n \geq 0$, are finitely generated with no elements of finite order, $G/A^{-n}H$, $n \geq 0$, are finite-dimensional tori. We have $\bigcup_{n=0}^{\infty} A^n Y_m = \Gamma$, $\bigcap_{n=0}^{\infty} A^{-n}H = \{0\}$ and $\zeta(H) \leq \zeta(A^{-1}H) \leq \dots$, $\bigvee_{n=0}^{\infty} \zeta(A^{-n}H) = \epsilon$. Since $(T^{-1})_{\zeta(A^{-n}H)}$ is an ergodic affine transformation of $G/A^{-n}H$ we have $\pi(T) = \pi(T^{-1}) = \bigvee_{n=0}^{\infty} \pi'((T^{-1})_{\zeta(A^{-n}H)})$ by Theorem 1, and if $\Gamma'_k = \{\gamma \in \Gamma \mid B^k \gamma = 0\}$ where $B = A - I$, then by part (i)

$$\pi'((T^{-1})_{\zeta(A^{-n}H)}) = \zeta\left(\text{ann}\left(\bigcup_{k=1}^{\infty} \Gamma'_k \cap A^n Y_m\right)\right) \quad (\text{see §1.3}).$$

Therefore

$$\begin{aligned}\pi(T) &= \bigvee_{n=0}^{\infty} \zeta \left(\text{ann} \left(\bigcup_{k=1}^{\infty} \Gamma'_k \cap A^n Y_m \right) \right) \\ &= \zeta \left(\text{ann} \left(\bigcup_{k=1}^{\infty} \Gamma'_k \right) \right) = \eta(T).\end{aligned}$$

(iii) Suppose now that G is a general compact connected metric abelian group. Γ is countable, $\Gamma = \{\gamma_1, \gamma_2, \dots\}$. Let Y_n be the smallest subgroup of Γ containing $\gamma_1, \dots, \gamma_n$ and invariant relative to A (i.e. $A Y_n \subset Y_n$). If $H_n = \text{ann}(Y_n)$, $A H_n \subset H_n$ and $T^{-1} \zeta(H_n) \leq \zeta(H_n)$. We have

$$\begin{aligned}Y_1 \subset Y_2 \subset Y_3 \subset \dots, & \quad \bigcup_{n=1}^{\infty} Y_n = \Gamma \\ H_1 \supset H_2 \supset H_3 \supset \dots, & \quad \bigcap_{n=1}^{\infty} H_n = \{0\},\end{aligned}$$

and

$$\zeta(H_1) \leq \zeta(H_2) \leq \zeta(H_3) \leq \dots, \quad \bigvee_{n=1}^{\infty} \zeta(H_n) = \epsilon.$$

Using the fact that Y_n is finitely generated with respect to the endomorphism $A_{\Gamma(H_n)}$ we have $\pi'(T_{\zeta(H_n)}) = \zeta(\text{ann}(\bigcup_{k=1}^{\infty} \Gamma'_k \cap Y_n))$ by part (ii) where $\Gamma'_k = \{\gamma \in \Gamma \mid B^k \gamma = 0\}$, $B = A - I$. (See §1.3.) Therefore by Theorem 1

$$\begin{aligned}\pi(T) &= \bigvee_n \pi'(T_{\zeta(H_n)}) = \bigvee_n \zeta \left(\text{ann} \left(\bigcup_{k=1}^{\infty} \Gamma'_k \cap Y_n \right) \right) \\ &= \zeta \left(\text{ann} \left(\bigcup_{k=1}^{\infty} \Gamma'_k \right) \right) = \eta(T).\end{aligned}$$

This completes the proof of the theorem.

REMARK. To prove part (ii) we could have used a result of Laxton and Parry [5], which states that an automorphism A of a compact connected metric abelian group has a dense set of periodic points if the character group is finitely generated with respect to A .

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