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ON TYPE I C^*-ALGEBRAS

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1. **Introduction.** Recently, the author [4] proved the equivalence of type I C^*-algebras and GCR C^*-algebras without the assumption of separability. On the other hand, for separable type I C^*-algebras, we have a simpler criterion as follows: a separable C^*-algebra $\mathfrak{A}$ is of type I if and only if every irreducible image contains a nonzero compact operator.

It has been open whether or not this remains true when $\mathfrak{A}$ is not separable (cf. [1], [2], [3]).

In the present paper, we shall show that a C^*-algebra $\mathfrak{A}$ is GCR if and only if every irreducible image contains a nonzero compact operator, so that by the author's previous theorem [4], the above problem is affirmative for arbitrary C^*-algebra.

2. **Theorem.** In this section, we shall show the following theorem.

Theorem. *A C^*-algebra $\mathfrak{A}$ is of type* I *if and only if every irreducible image contains a nonzero compact operator.*

Proof. Suppose that a C^*-algebra $\mathfrak{A}$ is of type I, then it is GCR and so every irreducible image contains a nonzero compact operator (cf. [1], [2], [3], [4]).

Conversely suppose that every irreducible image of $\mathfrak{A}$ contains a nonzero compact operator. It is enough to assume that $\mathfrak{A}$ has the unit I. We shall assume that $\mathfrak{A}$ is not of type I. Then it is not GCR; then there is a separable nontype I C^*-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ (cf. [2], [4]). Take a pure state ϕ on $\mathfrak{B}$ such that the image of $\mathfrak{B}$ under the irreducible *-representation $\{U_\phi, \mathfrak{H}_\phi\}$ of $\mathfrak{B}$ constructed via ϕ does not contain any nonzero compact operator, where $\mathfrak{H}_\phi$ is a Hilbert space.

Received by the editors September 12, 1966.

[1] This paper was written with partial support from ONR Contract NR-551(57).

Let $\mathcal{E}$ be the set of all pure states ψ on $\mathfrak{A}$ such that $\psi=\phi$ on $\mathfrak{B}$. We shall define a partial ordering $\prec$ in $\mathcal{E}$ in the following. Take $\psi\in\mathcal{E}$, and $\{\pi_\psi, \mathfrak{H}_\psi\}$ be the irreducible *-representation of $\mathfrak{A}$ constructed via ψ, then $\pi_\psi(\mathfrak{A})$ contains a nonzero compact operator; hence $\pi_\psi(\mathfrak{A})$ contains the algebra $C(\mathfrak{H}_\psi)$ of all compact operators (cf. [1]). Let $\mathfrak{D}(\psi) =\pi_\psi^{-1}(C(\mathfrak{H}_\psi))$, then $\mathfrak{D}(\psi)$ is an ideal of $\mathfrak{A}$. For $\psi_1, \psi_2\in\mathcal{E}$, we shall define the order as follows: $\psi_1\prec\psi_2$ if $\mathfrak{D}(\psi_1)\subset\mathfrak{D}(\psi_2)$. Let $\{\psi_\alpha|\alpha\in\Pi\}$ be a linearly ordered subset of $\mathcal{E}$, and let $\mathfrak{D}$ be the uniform closure of $\bigcup_{\alpha\in\Pi}\mathfrak{D}(\psi_\alpha)$, then $\mathfrak{D}$ is an ideal of $\mathfrak{A}$. Let $\mathfrak{F}$ be the kernel of the representation $\{\overline{U}_\phi, \mathfrak{H}_\phi\}$ of $\mathfrak{B}$. First of all we shall show that $\mathfrak{B}\cap\mathfrak{D}\subset\mathfrak{F}$. Suppose that $\mathfrak{B}\cap\mathfrak{D}\not\subset\mathfrak{F}$, then there is an element $b\in(\mathfrak{B}\cap\mathfrak{D})\cap\mathfrak{F}^c$ and $b_n\in\mathfrak{D}(\psi_{\alpha n})$ for $n=1, 2, 3, \cdots$ such that $\|U_\phi(b)\|=1$ and $\|b-b_n\|<1/n$ for $n=1, 2, 3, \cdots$, where $\mathfrak{F}^c$ is the complement of $\mathfrak{F}$ in $\mathfrak{B}$.

Take the representation $\{\pi_{\psi_{\alpha n}}, \mathfrak{H}_{\psi_{\alpha n}}\}$ of $\mathfrak{A}$, then $\|\pi_{\psi_{\alpha n}}(b)-\pi_{\psi_{\alpha n}}(b_n)\| <1/n$.

Let $[\pi_{\psi_{\alpha n}}(\mathfrak{B})I_{\psi_{\alpha n}}]$ be the closed subspace generated by $\pi_{\psi_{\alpha n}}(\mathfrak{B})I_{\psi_{\alpha n}}$, where $I_{\psi_{\alpha n}}$ is the image of I in $\mathfrak{H}_{\psi_{\alpha n}}$, and let E_n' be the orthogonal projection of $\mathfrak{H}_{\psi_{\alpha n}}$ onto $[\pi_{\psi_{\alpha n}}(\mathfrak{B})I_{\psi_{\alpha n}}]$. Then the representation $y\to\pi_{\psi_{\alpha n}}(y)E_n'$ for $y\in\mathfrak{B}$ is equivalent to $\{U_\phi, \mathfrak{H}_\phi\}$.

On the other hand, $\|E_n'\pi_{\psi_{\alpha n}}(b)E_n'-E_n'\pi_{\psi_{\alpha n}}(b_n)E_n'\|<1/n$, and $E_n'\pi_{\psi_{\alpha n}}(b_n)E_n'$ is a compact operator on $E_n'\mathfrak{H}_{\psi_{\alpha n}}$. Hence, there is a compact operator T_n on $\mathfrak{H}_\phi$ such that $\|U_\phi(b)-T_n\|<1/n$, because $E_n'\pi_{\psi_{\alpha n}}(b)E_n'=\pi_{\psi_{\alpha n}}(b)E_n'$. Therefore, $U_\phi(b)$ is a nonzero compact operator on $\mathfrak{H}_\phi$; this is a contradiction and so $\mathfrak{B}\cap\mathfrak{D}\subset\mathfrak{F}$.

Next, let us consider a C^*-algebra $\mathfrak{A}/\mathfrak{D}$, then $\mathfrak{B}+\mathfrak{D}/\mathfrak{D}$ is a C^*-subalgebra of $\mathfrak{A}/\mathfrak{D}$, because every *-homomorphic image of a C^*-algebra into another C^*-algebra is closed and the mapping $x\to x+\mathfrak{D}(x\in\mathfrak{B})$ of $\mathfrak{B}$ into $\mathfrak{A}/\mathfrak{D}$ is *-homomorphic.

The state ϕ on $\mathfrak{B}$ can be canonically considered a pure state on $\mathfrak{B}+\mathfrak{D}/\mathfrak{D}$, because $\mathfrak{B}\cap\mathfrak{D}\subset\mathfrak{F}$ and the C^*-algebra $\mathfrak{B}+\mathfrak{D}/\mathfrak{D}$ is *-isomorphic to the C^*-algebra $\mathfrak{B}/\mathfrak{B}\cap\mathfrak{D}$. Take a pure state extension $\tilde{\phi}$ of ϕ to $\mathfrak{A}/\mathfrak{D}$, then we can define a pure state ψ of $\mathfrak{A}$ by $\psi(y)=\tilde{\phi}(y+\mathfrak{D})$ for $y\in\mathfrak{A}$. Then we have $\psi=\phi$ on $\mathfrak{B}$ and so $\psi\in\mathcal{E}$.

Clearly $\mathfrak{D}(\psi_\alpha)\subset\mathfrak{D}(\chi)$; hence $\psi_\alpha\prec\psi$, and so by Zorn's lemma $\mathcal{E}$ contains a maximal element ψ_0.

Now we shall show $\mathfrak{D}(\psi_0)\cap\mathfrak{B}\not\subset\mathfrak{F}$. Assume that $\mathfrak{D}(\psi_0)\cap\mathfrak{B}\subset\mathfrak{F}$, then by the analogous discussion with the above, ϕ can be canonically considered a pure state on a C^*-subalgebra $\mathfrak{B}+\mathfrak{D}(\psi_0)/\mathfrak{D}(\psi_0)$ of $\mathfrak{A}/\mathfrak{D}(\psi_0)$; therefore we can have a pure state ψ_β on $\mathfrak{A}$ such that $\psi_\beta(\mathfrak{D}(\psi_0))=0$ and $\psi_\beta=\phi$ on $\mathfrak{B}$; hence $\mathfrak{D}(\psi_\beta)\supsetneqq\mathfrak{D}(\psi_0)$, a contradiction.

On the other hand, $\mathfrak{D}(\psi_0)\cap\mathfrak{B}\not\subset\mathfrak{F}$ also implies a contradiction, because $\pi_{\psi_0}(b)$ is a compact operator on $\mathfrak{H}_{\psi_0}$ for some $b \in (\mathfrak{D}(\psi_0) \cap \mathfrak{B}) \cap \mathfrak{F}^c$; hence $\pi_{\psi_0}(b)E'$ is compact, where E' is the orthogonal pro˙ection of $\mathfrak{H}_{\psi_0}$ onto $[\pi_{\psi_0}(\mathfrak{B})I_{\psi_0}]$; hence $U_\phi(b)=0$ and so $b \in \mathfrak{F}$.

Hence we can conclude that $\mathfrak{A}$ is of type I. This completes the proof.

References

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