ANOTHER THEOREM ON BOUNDED ANALYTIC FUNCTIONS

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This note is an attempt to solve the conjecture stated at the end of the preceding paper [1]. We are able to prove the following.

THEOREM 1. *Let* $\{\phi_n\}$ *be a sequence of summable functions on the circle such that*

$$l(f) = \lim_{n\to\infty} \int f\phi_n$$

exists for all $f \in H^\infty$ *(space of bounded functions on the circle with a positive spectrum; the integral is taken over the circle). Then there is a* $\phi \in L^1$ *such that*

$$l(f) = \int f\phi$$

for all $f \in A$ *(space of continuous functions on the circle with a positive spectrum).*

PROOF. As in [1] we see first that there exists a measure $d\mu$ on the circle such that $l(f) = \int f d\mu$ whenever $f \in A$. Let us prove that $d\mu$ is absolutely continuous.

Suppose that $d\mu$ is not absolutely continuous. Let E be a closed set on the circle, with Lebesgue measure zero, such that $\int_E d\mu = \mu(E) \neq 0$. Let h be a function in A such that $h=1$ on E and $|h| < 1$ outside (the existence of such a function is well known; it is used also in [1]). We have the following equalities ($m=1, 2, \cdots; n=1, 2, \cdots$):

(1) $\lim_{m\to\infty} \int h^m d\mu = \mu(E)$,

(2) $\lim_{m\to\infty} \int h^m \phi_n = 0$ for all n's,

(3) $\lim_{n\to\infty} \int h^m \phi_n = \int h^m d\mu$ for all m's.

If the sequence m_j is rapidly increasing (meaning that m_{j+1} is sufficiently large when m_j is given), we have

$$f = \sum_{j=1}^{\infty} (-1)^j h^{m_j} \in H^\infty.$$

For, given m_j, we define E_j as the set where $|h^{m_j} - 1| < 2^{-j}$, and we have $|h^{m_{j+1}}| < 2^{-j}$ on $\complement E_j$ when m_{j+1} is large enough. We shall write L_1 for this condition on the m_j.

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We shall define by induction two sequences m_j (satisfying L_1) and n_j. We shall use the formula

$$\int f\phi_{n_j} = \sum_{k=1}^{j-1} (-1)^k \int h^{m_k}\phi_{n_j} + (-1)^j \int h^{m_j}\phi_{n_j} + \sum_{k=j+1}^{\infty} (-1)^k \int h^{m_k}\phi_{n_j}$$
$$= A_j + B_j + C_j.$$

We write L_2 for the condition

$$\sum_{k=j+1}^{\infty} \left| \int h^{m_k}\phi_{n_j} \right| < \frac{1}{12} \mid \mu(E) \mid ;$$

by (2), it is satisfied when $m_{j+1}, m_{j+2}, \cdots$ are chosen large enough, n_j being given. We write L_3 for the condition

$$\left| \int h^{m_j} d\mu \right| > \frac{11}{12} \mid \mu(E) \mid ;$$

by (1), it is satisfied when m_j is large. Now suppose that $m_1, \cdots, m_{j-1}, n_1, \cdots, n_{j-1}$ are given in such a manner that the conditions L_1, L_2, L_3 are satisfied at this stage. They will be satisfied at the following stage if m_j is sufficiently large, $m_j \geqq m_j^*$, say. We define n_j^* so that $n \geqq n_j^*$ implies

$$\mid A_j - A_j^\infty \mid \; < \; \mid \mu(E) \mid /12,$$

where

$$A_j^\infty = \sum_{k=1}^{j-1} (-1)^k \int h^{m_k} d\mu;$$

that is possible because of (3). Now we consider two cases, namely

(α) $\mid A_{j-1}+B_{j-1}-A_j^\infty \mid \; \leqq 5 \mid \mu(E) \mid /12,$

(β) $\mid A_{j-1}+B_{j-1}-A_j^\infty \mid \; > 5 \mid \mu(E) \mid /12.$

In the case (α), we choose $m_j = m_j^*$, and n_j large enough ($\geqq n_j^*$) so that $|B_j| > 11|\mu(E)|/12$; that is possible because of (3) and L_3. In the case (β), we choose $n_j = n_j^*$, and m_j large enough ($\geqq m_j^*$) so that $|B_j| < |\mu(E)|/12$; that is possible because of (2). In each case, we have

$$\mid A_{j-1} + B_{j-1} - A_j - B_j \mid \; > 3 \mid \mu(E) \mid /12.$$

Taking L_2 into account, we have $|C_{j-1}|$ and $|C_j|$ majorized by $|\mu(E)|/12$, and therefore

$$\left| \int f\phi_{n_{j-1}} - \int f\phi_{n_j} \right| > \frac{1}{12} \mid \mu(E) \mid .$$

Therefore the sequence $\int f\phi_n$ is not convergent, against our assumption. The contradiction proves that $d\mu$ is absolutely continuous, that is $l(f)=\int f\,d\mu=\int f\phi$ whenever $f\in A$, for some $\phi\in L^1$.

REMARK. If $\phi_n(t)=\sum_{k=-\infty}^{\infty}a_{n,k}e^{-ikt}$, the assumption of the theorem is the existence of $\lim_{n\to\infty}\sum_0^\infty a_{n,k}b_k$ for all $\sum_0^\infty b_k\,e^{ikt}\in H^\infty$. The conclusion is $\lim_{n\to\infty}a_{n,k}=\int\phi(\theta)e^{ik\theta}$ for some $\phi\in L^1$ $(k=0, 1, 2, \cdots)$. Theorem 1 of [1] follows as a particular case.

We are not able to prove that $l(f)=\int f\phi$ for all $f\in H^\infty$. Nevertheless, this holds for many functions in H^∞. Precisely, we have

THEOREM 2. *Keeping the same notations as in Theorem* 1, *let D_l be the set of all $f\in H^\infty$ such that $l(f)=\int f\phi$, and let D be the intersection of the D_l for all l. Then* (α) *D_l is a closed subspace of H^∞ and, given any $f\in H^\infty$, almost all translates of f belong to D_l.* (β) *D is a closed subalgebra of H^∞, invariant under translation; it contains all $f\in H^\infty$ such that $fg\in D$ for some outer function $g\in D$; in particular, it contains all $f\in H^\infty$ which are continuous on the circle except on a closed set of measure zero.*

PROOF. We may suppose that the ϕ_n are trigonometric polynomials. By the Banach-Steinhaus theorem, the linear functionals $f\to\int f\phi_n$ are uniformly bounded on A. There exist measures $d\mu_n$, with bounded norms, such that $\int f\phi_n=\int f\,d\mu_n$ for all $f\in A$. By the F. and M. Riesz theorem (or another device) the $d\mu_n$ are absolutely continuous. Therefore we may suppose that the ϕ_n have bounded L^1-norms.

In order to prove (α) we may suppose $\phi=0$. The fact that D_l is a closed subspace of H^∞ is obvious. Given $f\in H^\infty$, we write $f_s(t)=f(t-s)$. Given $\psi\in L^1$, we have $f*\psi\in A$, and by Theorem 1

$$\lim_{n\to\infty}\int\int\phi_n(t)f(t-s)\psi(s)dsdt=0.$$

By assumption

$$\lim_{n\to\infty}\int\phi_n(t)f(t-s)dt=l(f_s)$$

and since the ϕ_n have bounded L^1-norms, the integrals $\int\phi_n(t)f(t-s)dt$ are uniformly bounded with respect to n and s. By the Lebesgue theorem

$$\int l(f_s)\psi(s)ds=0$$

and since ψ is an arbitrary function in L^1, $l(f_s)=0$ for almost every s. That proves (α).

In order to prove (β) we write

$$\lim_{n\to\infty} \int fg\phi_n = l_f(g) = l_g(f) = l(fg) \qquad (f \in H^\infty,\ g \in H^\infty),$$

$$l_f(g) = \int g\phi_f \qquad \text{when } g \in D.$$

We have $A \subset D$ as a reformulation of Theorem 1.

Suppose $f \in A$. Taking $g \in A$, we have $fg \in A$. Since $fg \in D$ and $g \in D$, we have

$$\int fg\phi = l(fg) = l_f(g) = \int g\phi_f, \tag{4}$$

$$\int g(f\phi - \phi_f) = 0 \tag{5}$$

and since g is arbitrary in A, $f\phi = \phi_f$ (mod H_0') (meaning that the Fourier coefficients or order $\leqq 0$ are the same).

Now suppose $g \in D$. Taking $f \in A$ we have

$$l(fg) = l_f(g) = \int g\phi_f = \int fg\phi \tag{6}$$

since $f\phi = \phi_f$ (mod H_0'). Therefore $fg \in D_l$ and, l being arbitrary, $fg \in D$. Since $fg \in D$ and $g \in D$, we have

$$\int f(g\phi - \phi_g) = 0$$

and since f is arbitrary in A, $g\phi = \phi_g$ (mod H_0').

If $f \in D$ and $g \in D$, we still have (6) because $f\phi = \phi_f$ (mod H_0'), and $fg \in D$ as a consequence. Therefore D is a subalgebra of H^∞. It is closed because each D_l is closed, and it is obviously invariant under translation.

Finally, suppose that $f \in H^\infty$, $g \in D$, g is an outer function and $fg \in D$. We still have (4) and (5). Moreover, since D is an algebra, we have

$$\int gh(f\phi - \phi_f) = 0$$

for all $h \in D$. Therefore $g(f\phi - \phi_f) = 0$ (mod H_0'). Since g is an outer function, it follows that $f\phi = \phi_f$ (mod H_0'). As a conclusion

$$l(f) = l_f(1) = \int \phi_f = \int f\phi,$$

that is, $f \in D_l$, and since l is arbitrary, $f \in D$.

Given a closed set K of measure zero on the circle, there exists a continuous outer function g vanishing on K (that follows immediately from a proof of Fatou's theorem). If f is continuous except on K, $fg \in A$, therefore $f \in D$. That ends the proof of Theorem 2.

Reference

1. George Piranian, A. L. Shields and J. H. Wells, *Bounded analytic functions and absolutely continuous measures*, Proc. Amer. Math. Soc. **18** (1967), pp., 818-826.

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