A COMPLETION FOR A QUASI UNIFORM SPACE

RONALD A. STOLTENBERG

1. **Introduction.** In a recent publication Pervin and Sieber [2] left open the question of whether or not a quasi uniform space had a completion using their definitions. It is the purpose of this paper to answer this question.

The following definitions are as given by Pervin and Sieber [2].

1.1. **DEFINITION.** A filter $\mathcal{F}$ in a quasi uniform space $(X, \mathcal{U})$ *converges* to a point $x \in X$ if and only if for each $U \in \mathcal{U}$, $U(x) \in \mathcal{F}$.

1.2. **DEFINITION.** A filter $\mathcal{F}$ in a quasi uniform space $(X, \mathcal{U})$ is *Cauchy* if and only if for each $U \in \mathcal{U}$ there is $x \in X$ such that $U(x) \in \mathcal{F}$.

1.3. **DEFINITION.** A quasi uniform space $(X, \mathcal{U})$ is *complete* if and only if each Cauchy filter in X converges to a point of X .

A function f from a quasi uniform space $(X, \mathcal{U})$ to a quasi uniform space $(Y, \mathcal{V})$ is quasi uniformly continuous relative to $\mathcal{U}$ and $\mathcal{V}$ if and only if for each $V \in \mathcal{V}$ the set $\{(x, y): (f(x), f(y)) \in V\} \in \mathcal{U}$. f is a quasi uniform isomorphism if and only if f is one-to-one onto Y and f^{-1} is quasi uniformly continuous. A quasi uniform space $(X, \mathcal{U})$ can be embedded in a quasi uniform space $(Y, \mathcal{V})$ when there exists a quasi uniform isomorphism from $(X, \mathcal{U})$ onto a subspace of $(Y, \mathcal{V})$. These definitions are the same as those given in [3].

1.4. **DEFINITION.** A quasi uniform space $(X, \mathcal{U})$ has a *completion* if and only if $(X, \mathcal{U})$ can be embedded in a complete quasi uniform space $(X^*, \mathcal{U}^*)$ in such a way that X is a dense subset of X^* .

2. **Main results.** A quasi pseudometric d on a set X is a function from $X \times X$ into the set of positive real numbers such that for $x, y, z \in X$: (a) $d(x, x) = 0$ and (b) $d(x, y) \leq d(x, z) + d(z, y)$. The pair (X, d) is called a quasi pseudometric space. Compare Kelly [1]. It is easy to see that a quasi pseudometric space (X, d) is a quasi uniform space and that a filter $\mathcal{F}$ in (X, d) is Cauchy if and only if for each $\epsilon > 0$ there exists $x \in X$ such that $S_\epsilon(x) = \{y: d(x, y) < \epsilon\} \in \mathcal{F}$.

2.1. **THEOREM.** Every quasi pseudometric space (X, d) has a completion (X^*, d^*) which is a quasi pseudometric space.

PROOF. We may and do assume that $d(x, y) \leq 1$ for all $x, y \in X$. Let

$$\bar{X} = \{\bar{x}: \bar{x} = \{x_n: n \in N\} \text{ is a sequence in } X\}.$$

Define

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$$\bar{d}: \bar{X} \times \bar{X} \rightarrow R: \begin{cases} \bar{d}(\bar{x}, \bar{y}) = 0 \text{ if } \bar{y} \text{ is a subsequence of } \bar{x}. \\ \bar{d}(\bar{x}, \bar{y}) = \liminf_n \limsup_m d(x_n, y_m) \text{ otherwise.} \end{cases}$$

(a) LEMMA. Suppose $\bar{y}$ is not a subsequence of $\bar{x}$. If $\bar{d}(\bar{x}, \bar{y}) < \delta$ then for each $K > 0$ there exists $k_0 \geq K$ and $L(k_0)$ such that for $j \geq L(k_0)$, $d(x_{k_0}, y_j) < \delta$. On the other hand if for each $K > 0$ there is $k_0 \geq K$ and $L(k_0)$ such that $d(x_{k_0}, y_j) < \delta$ whenever $j \geq L(k_0)$ then $\bar{d}(\bar{x}, \bar{y}) \leq \delta$.

PROOF. Suppose $\bar{d}(\bar{x}, \bar{y}) < \delta$. Now $\bar{d}(\bar{x}, \bar{y}) = \liminf_n \limsup_m d(x_n, y_m) = U$. Now $U < \delta$ implies for every $K < 0$ there exists $k_0 \geq K$ such that $\limsup_m d(x_{k_0}, y_m) = U' < \delta$. $U' < \delta$ implies there exists $L(k_0)$ such that for $j \geq L(k_0)$, $d(x_{k_0}, y_j) < \delta$. So we have the first part of the lemma.

On the other hand suppose for each $K > 0$ there exists $k_0 \geq K$ and $L(k_0)$ such that for $j \geq L(k_0)$, $d(x_{k_0}, y_j) > \delta$. Hence $\limsup_m d(x_{k_0}, y_m) \leq \delta$ since for $j \geq L(k_0)$, $d(x_{k_0}, y_j) < \delta$. But now $\inf_{k \geq n} \limsup_m d(x_k, y_m) \leq \delta$ for any k so $\sup_n \inf_{k \geq n} \limsup_m d(x_k, y_m) = \liminf_n \limsup_m d(x_n, y_m) \leq \delta$.

(b) LEMMA. $\bar{d}$ is a quasi pseudometric for $\bar{X}$.

PROOF. Clearly $\bar{d}(\bar{x}, \bar{x}) = 0$ for all $\bar{x} \in \bar{X}$. Let $\bar{x}, \bar{y}, \bar{z} \in \bar{X}$. To show that $\bar{d}(\bar{x}, \bar{y}) \leq \bar{d}(\bar{x}, \bar{z}) + \bar{d}(\bar{z}, \bar{y})$ there are four cases to consider:

Case 1. $\bar{y}$ is a subsequence of $\bar{x}$.

In this case $\bar{d}(\bar{x}, \bar{y}) = 0 \leq \bar{d}(\bar{x}, \bar{z}) + \bar{d}(\bar{z}, \bar{y})$.

Case 2. $\bar{z}$ is a subsequence of $\bar{x}$ and $\bar{y}$ not a subsequence of $\bar{x}$.

In this case $\bar{d}(\bar{x}, \bar{z}) = 0$. Suppose $\bar{d}(\bar{z}, \bar{y}) < \bar{d}(\bar{x}, \bar{y})$. Let $\delta = \bar{d}(\bar{x}, \bar{y}) - \bar{d}(\bar{z}, \bar{y})$ now $\bar{d}(\bar{z}, \bar{y}) < \bar{d}(\bar{z}, \bar{y}) + \delta/2$. So for each $P > 0$ there is $p_0 \geq P$ and $L(p_0)$ such that $d(z_{p_0}, y_j) < \bar{d}(\bar{z}, \bar{y}) + \delta/2$ whenever $j \geq L(p_0)$. However $\bar{z}$ is a subsequence of $\bar{x}$. So $z_{p_0} = x_{n_{p_0}}$ where $n_{p_0} \geq p_0 \geq P$. So for each $P > 0$ there is $n_{p_0} \geq P$ and $L(n_{p_0}) = L(p_0)$ such that $d(x_{n_{p_0}}, y_j) < \bar{d}(\bar{z}, \bar{y}) + \delta/2$ for all $j \geq L(n_{p_0})$. So by Lemma (a) $\bar{d}(\bar{x}, \bar{y}) \leq \bar{d}(\bar{z}, \bar{y}) + \delta/2 < \bar{d}(\bar{x}, \bar{y})$, a contradiction. So $\bar{d}(\bar{x}, \bar{y}) \leq \bar{d}(\bar{x}, \bar{z}) + \bar{d}(\bar{z}, \bar{y})$.

Case 3. $\bar{y}$ is a subsequence of $\bar{z}$ and $\bar{y}$ is not a subsequence of $\bar{x}$.

In this case $\bar{d}(\bar{z}, \bar{y}) = 0$. Suppose $\bar{d}(\bar{x}, \bar{z}) < \bar{d}(\bar{x}, \bar{y})$. Let $\delta = \bar{d}(\bar{x}, \bar{y}) - \bar{d}(\bar{x}, \bar{z})$. Now $\bar{d}(\bar{x}, \bar{z}) < \bar{d}(\bar{x}, \bar{z}) + \delta/2$. So for each $P > 0$ there is $p_0 \geq P$ and $L(p_0)$ such that $d(x_{p_0}, z_j) < \bar{d}(\bar{x}, \bar{z}) + \delta/2$ for all $j \geq L(p_0)$. However $\bar{y}$ is a subsequence of $\bar{z}$. So $y_k = z_{j_k}$. Now for $j_k \geq L(p_0)$, $d(x_{p_0}, z_{j_k}) = d(x_{p_0}, y_k) < \bar{d}(\bar{x}, \bar{z}) + \delta/2$. Let $L'(p_0)$ be the smallest integer such that $j_{L'(p_0)} \geq L(p_0)$. Thus for each $P > 0$ there is $p_0 \geq P$ and $L'(p_0)$ such that $d(x_{p_0}, y_k) < \bar{d}(\bar{x}, \bar{z}) + \delta/2$ for all $k \geq L'(p_0)$. So $\bar{d}(\bar{x}, \bar{y}) \leq \bar{d}(\bar{x}, \bar{z}) + \delta/2 < \bar{d}(\bar{x}, \bar{y})$, a contradiction. So $\bar{d}(\bar{x}, \bar{y}) \leq \bar{d}(\bar{x}, \bar{z}) + \bar{d}(\bar{z}, \bar{y})$.

Case 4. $\bar{z}$ is not a subsequence of $\bar{x}$, $\bar{y}$ is not a subsequence of $\bar{z}$ and $\bar{y}$ is not a subsequence of $\bar{x}$.

In this case $\bar{d}(\bar{x}, \bar{y}) = \liminf_n \limsup_m d(x_n, y_m)$, $\bar{d}(\bar{x}, \bar{z}) = \liminf_n \limsup_m d(x_n, z_m)$ and $\bar{d}(\bar{z}, \bar{y}) = \liminf_n \limsup_m d(z_n, y_m)$. Let $\delta > 0$; let $U_1 = \bar{d}(\bar{x}, \bar{z})$; let $U_2 = \bar{d}(\bar{z}, \bar{y})$. Let $K > 0$.

(1) $U_1 < U_1 + \delta/2$ implies there exists $k_0 \geq K$ and $L(k_0)$ such that for $j \geq L(k_0)$, $d(x_{k_0}, z_j) < U_1 + \delta/2$.

(2) $U_2 < U_2 + \delta/2$ implies there exists $j_0 \geq L(k_0)$ and $L(j_0)$ such that for $s \geq L(j_0)$, $d(z_{j_0}, y_s) < U_2 + \delta/2$.

Now $d(x_{k_0}, y_s) \leq d(x_{k_0}, z_{j_0}) + d(z_{j_0}, y_s) < U_1 + U_2 + \delta$. So for each $K > 0$ there exists $k_0 \geq K$ and $L'(k_0) = L(j_0)$ such that for $s \geq L'(k_0)$, $d(x_s, y_{k_0}) < U_1 + U_2 + \delta$. Hence by Lemma (a) $\bar{d}(\bar{x}, \bar{y}) \leq U_1 + U_2 = \bar{d}(\bar{x}, \bar{z}) + \bar{d}(\bar{z}, \bar{y})$.

It follows that $\bar{d}$ is a quasi pseudometric on X .

(c) LEMMA. $(\bar{X}, \bar{d})$ is a complete quasi pseudometric space.

PROOF. Let $\mathcal{F}$ be a Cauchy filter in $(\bar{X}, \bar{d})$. Then for each integer k there is $\bar{x}^k \in \bar{X}$ such that $S_k(\bar{x}^k) = \{\bar{y} : \bar{d}(\bar{x}^k, \bar{y}) < \frac{1}{2}k\} \in \mathcal{F}$. Now $\bar{x}^k = \{x_j^k : j \in N\}$ for each k . Let $\bar{x} = \{x_1^1, x_1^2, x_1^3, x_2^1, x_2^2, x_3^1, \dots, x_1^k, x_2^{k-1}, x_3^{k-2}, \dots, x_k^1, \dots\} = \{x_{m(j, h-j)} : j = 1, 2, \dots, h-1 : h = 2, 3, \dots\}$ where $x_{m(j, h-j)} = x_j^{h-j}$ and $m(j, h-j) = ((h-2)(h-1))/2 + j$. Clearly $\bar{x}^k$ is a subsequence of $\bar{x}$ for each k . So $\bar{d}(\bar{x}, \bar{y}) \leq \bar{d}(\bar{x}, \bar{x}^k) + \bar{d}(\bar{x}^k, \bar{y}) < \frac{1}{2}k$ whenever $\bar{y} \in S_k(\bar{x}^k)$. So $S_k(\bar{x}^k) \subseteq S_k(\bar{x})$. Hence $S_k(\bar{x}) \in \mathcal{F}$ for each k . Thus $\mathcal{F}$ converges to $\bar{x}$ in $\bar{X}$. So $(\bar{X}, \bar{d})$ is complete.

(d) LEMMA. (X, d) can be embedded in $(\bar{X}, \bar{d})$.

PROOF. Let $\phi : (X, d) \rightarrow (\bar{X}, \bar{d}) : \phi(x) = \{x_n : n \in N\}$ where $x_n = x$ for all $n \in N$. $\bar{d}(\phi(x), \phi(y)) = \liminf_n \limsup_m d(x_n, y_m) = d(x, y)$; thus it follows that ϕ is a quasi uniform isomorphism from (X, d) onto a subspace denoted by X of $(\bar{X}, \bar{d})$. Hence (X, d) can be embedded in $(\bar{X}, \bar{d})$.

Let X^* be the closure of X in $(\bar{X}, \bar{d})$ and let $d^* = \bar{d}|_{X^*}$. Clearly (X^*, d^*) is a completion for (X, d) . This completes the proof of 2.1.

2.2. PROPOSITION. A closed subspace of a complete quasi uniform space is complete.

2.3. PROPOSITION. Let $(X_\alpha^*, \mathfrak{U}_\alpha)$ be a quasi uniform space for each α in an index set Ω . Let $\times_{\alpha \in \Omega} \{X_\alpha\} = X$ and let the product quasi uniformity be $\mathfrak{U}$. Then $(X, \mathfrak{U})$ is complete if and only if $(X_\alpha, \mathfrak{U}_\alpha)$ is complete for each $\alpha \in \Omega$.

The proofs of Propositions 2.2 and 2.3 are clear and are omitted.

2.4. THEOREM. *Every quasi uniform space $(X, \mathfrak{U})$ has a completion $(X^*, \mathfrak{U}^*)$.*

PROOF. By a theorem in [3] we have that $(X, \mathfrak{U})$ can be embedded in a product of quasi pseudometric space $\times \{(X_\alpha, d_\alpha) : \alpha \in \Omega\}$. We may and do assume $d_\alpha(x_\alpha, y_\alpha) \leq 1$ for all $x_\alpha, y_\alpha \in X_\alpha$ and for each $\alpha \in \Omega$. Each space (X_α, d_α) has a completion (X_α^*, d_α^*) . Clearly $(X, \mathfrak{U})$ can be embedded in $\times \{(X_\alpha^*, d_\alpha^*) : \alpha \in \Omega\}$. Let X^* be the closure of X in $\times \{(X_\alpha^*, d_\alpha^*) : \alpha \in \Omega\}$. Let $\mathfrak{U}^* = \{U \cap X^* \times X^* : U \text{ is a member of the product quasi uniformity for } \times \{(X_\alpha^*, d_\alpha^*) : \alpha \in \Omega\}\}$. From Propositions 2.2 and 2.3 it follows that $(X^*, \mathfrak{U}^*)$ is a completion for $(X, \mathfrak{U})$.

REFERENCES

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WASHINGTON STATE UNIVERSITY