THE ISOMORPHIC REFINEMENT THEOREM FOR DIRECT SUMS OF CLOSED GROUPS¹

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The purpose of this paper is to prove that if the primary group G is a direct sum of closed groups, then any two direct decompositions of G have isomorphic refinements. Special cases of this theorem have been proven by Kulikov [10], Kolettis [9], Enochs [1], and Irwin, Richman and Walker [8]. A special case due to Kulikov is now considered classical: If the primary group G is a direct sum of cyclic groups, then any two direct decompositions of G have isomorphic refinements. This result yields rather immediately the following, also due to Kulikov [10]. If G is a closed group, then any two direct decompositions of G admit isomorphic refinements. In [9] Kolettis embedded the two theorems of Kulikov into a slightly more general result. Enochs proved in [8] that any two direct decompositions of a primary group G into closed groups have isomorphic refinements, but it was not known heretofore that a direct summand of a direct sum of closed groups is always itself a direct sum of closed groups. However, it was shown in [1] that a direct summand of a direct sum of a countable number of closed groups is necessarily a direct sum of closed groups, and the general isomorphic refinement theorem was established for a group G that can be expressed as a direct sum of a countable number of a closed group. As indicated above, in this paper we shall remove the countability hypothesis.

We follow, for the most part, the notation and terminology of [2]. In particular, we use the term "closed" instead of "torsion complete."

LEMMA 1. *Suppose that H is a closed group and that $G = \sum_{\lambda \in \Lambda} G_\lambda$ is a direct sum of closed groups G_λ . If ϕ is a homomorphism from $H[p]$ into G that does not decrease heights (computed in H and G), then there is a positive integer n and a finite subset M of Λ such that $\phi(p^n H \cap H[p]) \subseteq \sum_{\lambda \in M} G_\lambda$.*

PROOF. Although the proof of Lemma 1 requires only a slight modification of the proof of similar but weaker results in [1], [5] and [8], we include an outline of the proof for completeness. If the lemma were false, we could choose $x_n \in (p^k H \cap H[p])$ such that:

(1) $p^k G \cap \{\pi_\lambda \phi(x_1), \pi_\lambda \phi(x_2), \dots, \pi_\lambda \phi(x_{n-1})\}_{\lambda \in \Lambda} = 0$ where π_λ is the natural projection of G onto G_λ .

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(2) $k \geq n$, and

(3) $\phi(x_n) \in \sum_{\lambda \in M} G_\lambda$ where M is the minimal subset of Λ such that $\{\phi(x_1), \phi(x_2), \dots, \phi(x_{n-1})\} \subseteq \sum_{\lambda \in M} G_\lambda$. Now $\sum x_n$ converges in H , but $\sum \phi(x_n)$ does not converge in G ; hence ϕ is discontinuous (in the p -adic topology) but this is impossible since ϕ does not decrease heights.

The next lemma states a rather immediate corollary of the preceding one.

LEMMA 2. *Suppose that the primary group H is a closed group and that $G = \sum \{g_\lambda\}$ is a direct sum of cyclic groups. If ϕ is a homomorphism from $H[p]$ into G that does not decrease heights, then there is a positive integer n such that $\phi(p^n H \cap H[p]) = 0$.*

Lemma 1 and Lemma 2 combine readily to yield the following result, which is what we actually need in the sequel.

LEMMA 3. *Suppose that H is a closed group and that $G = B + \sum_{\lambda \in \Lambda} G_\lambda$ where B is a direct sum of cyclic groups and G_λ is closed for each $\lambda \in \Lambda$. If ϕ is a homomorphism from $H[p]$ into G that does not decrease heights, then there is a positive integer n and a finite subset M of Λ such that $\phi(p^n H \cap H[p]) \subseteq \sum_{\lambda \in M} G_\lambda$.*

By a projection of a group G , we mean an idempotent endomorphism of G . It is an elementary and well-known result that a subgroup H of an abelian group G is a direct summand of G if and only if $H = \phi(G)$ is the image of some projection ϕ of G . Moreover, if a subsocle S of the primary group G supports a direct summand of G , then S must be the image of a projection of $G[p]$ that does not decrease heights; however, it follows from [3] and [11] that the converse does not hold. That the converse is valid for pure-complete groups G is the content of Theorem 1. Recall that a pure-complete group is a primary group with the property that each subsocle supports a pure subgroup; see [4] and [7].

THEOREM 1. *If the primary group G is pure-complete and if ϕ is a projection of $G[p]$ that does not decrease heights, then the image of ϕ supports a direct summand of G .*

PROOF. Let H be a pure subgroup of G such that $H[p] = \phi(G[p])$ and let S be the subsocle of $G+H$ defined by $S = \{(x, \phi(x)) : x \in G[p]\}$. The subgroup $K = \{(h, h) : h \in H\}$ is a pure subgroup of $G+H$. Since $(G+H)/K$ is isomorphic to G , it is pure-complete. Let L/K be a pure subgroup of $(G+H)/K$ such that $(L/K)[p] = \{S, K\}/K$. Then L is a pure subgroup of $G+H$ and $L[p] = S$. Since $L[p] = S$, it is clear that

L , as a subgroup of $G+H$, has trivial intersection with H . Since ϕ does not decrease heights and since L is pure, we can show by induction on n that $G[p^n] \subseteq \{L, H\}$. Thus $G \subseteq \{L, H\}$ and the natural projection of $G+H$ onto H is a projection of G onto H when restricted to L if G and L are identified in the obvious way.

COROLLARY 1. *If the primary group G is a direct sum of a countable number of closed groups and if S is a subsocle of G , then S supports a direct summand of G if and only if S is the image of a projection of $G[p]$ that does not decrease heights.*

PROOF. A direct sum of a countable number of closed groups is pure-complete by Theorem 5.6 in [7].

Since a primary group G is a direct sum of a countable number of closed groups if it is the direct sum of any number of cyclic groups, we have

COROLLARY 2. *If the primary group G is a direct sum of cyclic groups and if S is a subsocle of G , then S supports a direct summand of G if and only if S is the image of a projection of $G[p]$ that does not decrease heights.*

The next theorem is of a technical nature, but our main result rests heavily upon it.

THEOREM 2. *For the primary group G , suppose that $G = \sum_{\lambda \in \Lambda} G_\lambda$ where G_λ is closed for each $\lambda \in \Lambda$. Suppose that ϕ is a projection of $G[p]$ that does not decrease heights. Then $G = B + \sum_{\mu \in M} H_\mu$ where B is a direct sum of cyclic groups, $M \subseteq \Lambda$, H_μ is closed for each $\mu \in M$, $\phi(H_\mu[p]) \subseteq \sum_{\nu \in N} H_\nu$ for some finite subset $N = N_\mu$ of M , and $\phi(B[p]) \subseteq B$.*

PROOF. The theorem is trivial in case Λ is finite; take $B=0$, $M=\Lambda$, $H_\lambda=G_\lambda$. In order to free ourselves from a few technicalities, we shall assume that Λ is uncountable and, in the end, observe that with an obvious modification the proof covers the countable case as well.

Choose the index set Λ to be the initial segment of ordinal numbers less than Ω where Ω is the smallest ordinal of cardinality $m > \aleph_0$. Suppose that $\gamma \in \Lambda$ and that, for each $\beta < \gamma$, we have chosen a subset I_β of Λ and subset M_β of I_β such that:

(1) $\sum_{I_\beta} G_\lambda = \sum_{M_\beta} B_\mu + \sum_{M_\beta} H_\mu$, where B_μ is a direct sum of cyclic groups and H_μ is closed for each $\mu \in M_\beta$.

(2) If $\alpha < \beta < \gamma$, then $I_\alpha \subseteq I_\beta$ and $M_\alpha \subseteq M_\beta$. In the decomposition (1), B_μ and H_μ are independent of β .

(3) For each $\mu \in M_\beta$, $\phi(H_\mu[p]) \subseteq \sum_N H_\nu$ for some finite subset N of M_β .

$$(4) \beta \in I_{\beta+1}.$$

$$(5) |I_\beta| \leq \aleph_0 |\beta| \text{ and } I_0 = \emptyset.$$

If γ is a limit ordinal, define $I_\gamma = \bigcup_{\beta < \gamma} I_\beta$ and $M_\gamma = \bigcup_{\beta < \gamma} M_\beta$. Then conditions (1)–(5) hold for $\beta \leq \gamma$.

Assume that $\gamma - 1$ is defined. Suppose that n is a positive integer and that $\emptyset = J_0 \subset J_1 \subset \cdots \subset J_n$ is a strictly ascending sequence of finite subsets of Λ such that the following conditions hold. Set $K(i) = J_i - J_{i-1}$ for $i = 1, 2, \dots, n$.

$$(i) J_n \cap I_{\gamma-1} = \emptyset.$$

(ii) If $1 \leq i < n$, then $\sum_{K(i)} G_\gamma = B_{\mu(i)} + H_{\mu(i)}$ where $B_{\mu(i)}$ is bounded and $\mu(i)$ is the first element of $K(i)$.

(iii) If $1 \leq k < n - 1$, then $\phi(H_{\mu(k)}[p]) \subseteq \sum_F H_\mu + \sum_1^{k+1} H_{\mu(i)}$ where F is a finite subset of $M_{\gamma-1}$.

(iv) $\phi(H_{\mu(n-1)}[p]) \subseteq \sum_F H_\lambda + \sum_1^{n-1} H_{\mu(i)} + \sum_{K(n)} G_\lambda$ where F is a finite subset of $M_{\gamma-1}$.

Note that we may assume that $\gamma - 1 \in (I_{\gamma-1} \cup J_n)$. We wish to define a finite subset J_{n+1} of Λ in such a way that conditions (i)–(iv) remain valid when $n + 1$ replaces n . Throughout the remainder of the proof, we shall reserve the symbol $\sum_F$ to denote a finite summation over $M_{\gamma-1}$, that is, F is understood to denote an arbitrary (not fixed) finite subset of $M_{\gamma-1}$.

Since $K(n) = J_n - J_{n-1}$ is finite, $\sum_{K(n)} G_\lambda$ is closed. It follows from Lemma 3 that there is a finite set L of Λ and a decomposition

(I) $\sum_{K(n)} G_\lambda = B_{\mu(n)} + H_{\mu(n)}$ such that $B_{\mu(n)}$ is bounded, $L \cap (I_{\gamma-1} \cup J_n) = \emptyset$, and

$$(II) \phi(H_{\mu(n)}[p]) \subseteq \sum_F H_\mu + \sum_1^n H_{\mu(i)} + \sum_L G_\lambda.$$

By condition (5), we may assume that L is nonempty. And since $K(n)$ is nonempty, we can choose $\mu(n)$ to be the first element of $K(n)$; this is an admissible index for B and H in the sense that it has not been used before.

Now a crucial point of the proof is that the decomposition (I) can be made in such a way that it enjoys an additional property:

(III) $\phi(H_{\mu(n-1)}[p]) \subseteq \sum_F H_\mu + \sum_1^n H_{\mu(i)}$. We know, by condition (iv), that

$$(IV) \phi(H_{\mu(n-1)}[p]) \subseteq \sum_F H_\mu + \sum_1^{n-1} H_{\mu(i)} + B_{\mu(n)} + H_{\mu(n)}.$$

Since $B_{\mu(n)}$ is bounded, we can write $B_{\mu(n)} = A_{\mu(n)} + C_{\mu(n)}$ where $C_{\mu(n)}[p]$ is the image of $\phi(H_{\mu(n-1)}[p])$ under the natural projection of $\sum_{M_{\gamma-1}} H_\mu + \sum_1^{n-1} H_{\mu(i)} + B_{\mu(n)} + H_{\mu(n)}$ onto $B_{\mu(n)}$. Suppose that $x \in C_{\mu(n)}[p]$. For some $y \in H_{\mu(n-1)}[p]$,

$$\phi(y) = h + h_1 + h_2 + \cdots + h_{n-1} + x + h_n$$

where $h \in \sum_F H_\mu$ and $h_i \in H_{\mu(i)}$ for $i = 1, 2, \dots, n$. Since $\phi^2 = \phi$, we have

$$\phi(y - h_{n-1}) = \phi(h) + \phi(h_1) + \cdots + \phi(h_{n-2}) + \phi(x) + \phi(h_n).$$

Thus $\phi(x) \in \sum_F H_\mu + \sum_1^{n-1} H_{\mu(i)} + C_{\mu(n)} + H_{\mu(n)} + \sum_L G_\lambda$. It follows that

$$\phi((C_{\mu(n)} + H_{\mu(n)})[p]) \subseteq \sum_F H_\mu + \sum_1^{n-1} H_{\mu(i)} + C_{\mu(n)} + H_{\mu(n)} + \sum_L G_\lambda.$$

Thus we may replace $H_{\mu(n)}$ by $C_{\mu(n)} + H_{\mu(n)}$ and $B_{\mu(n)}$ by $A_{\mu(n)}$ and, in the process, gain condition (III).

Define $J_{n+1} = J_n \cup L$, and the induction is completed on the sequence $\{J_n\}$. Defining $I_\gamma = I_{\gamma-1} \cup (\bigcup_{n < \omega} J_n)$ and $M_\gamma = \{M_{\gamma-1}, \mu(n)\}_{n < \omega}$, we have at the same time completed the induction on the existence of the chains $\{I_\lambda\}$ and $\{M_\lambda\}$, $\lambda \in \Lambda$, with the desired properties (1)–(5). We remark that in case Λ is countable we could exhaust Λ , with the proper choice of J_n , in one step, that is, with I_1 and the argument simplifies.

Clearly, the I_λ 's and M_λ 's satisfying conditions (1)–(5) yield a decomposition $G = B + \sum_{\mu \in M} H_\mu$ that satisfies all the conditions of the theorem except, possibly, the condition $\phi(B[p]) \subseteq B$. Let ρ and σ denote the projections associated with the decomposition $G = B + \sum_{\mu \in M} H_\mu$, where $\rho(G) = B$. The endomorphism $\rho\phi$ of $B[p]$ is actually a projection of $B[p]$ that does not decrease heights. Corollary 2 implies that $\rho\phi(B[p])$ supports a direct summand of B . Let $B = B_0 + B_1$ where $B_1[p] = \rho\phi(B[p])$. Define $S_0 = (1 - \phi)(B_0[p])$ and $S_1 = (1 + \sigma\phi)B_1[p]$. There are pure subgroups B'_0 and B'_1 of G such that $B'_0[p] = S_0$, $B'_1[p] = S_1$, $\rho(B'_0 + B'_1) = B_0 + B_1$. Letting $B' = B'_0 + B'_1$, we have that $G = B' + \sum_{\mu \in M} H_\mu$. It is straightforward to show that $\phi(B'[p]) \subseteq B'$, and the proof of Theorem 2 is finished.

We are now prepared to prove a result which contains as a corollary the fact that a direct summand of a direct sum of closed groups is a direct sum of closed groups.

THEOREM 3. *Suppose that the primary group $G = \sum_{\lambda \in \Lambda} G_\lambda$ is a direct sum of closed groups G_λ . If ϕ is a projection of $G[p]$ that does not decrease heights, then $S = \phi(G[p])$ supports a direct summand of G . Furthermore, if H is any such direct summand of G , then $H = \sum_{\lambda \in \Lambda} H_\lambda$ where H_λ is a closed group.*

PROOF. The proof is by induction on the cardinality of Λ . If Λ is countable, Corollary 1 states that S supports a direct summand of G . A direct summand of G , in case Λ is countable, is a direct sum of a countable number of closed groups by Theorem 1 in [8]. Thus we may assume that Λ is uncountable. By Theorem 2, there is no loss of generality in assuming that, for each $\lambda \in \Lambda$, there is a countable sub-

set M of Λ such that $\phi(G_\lambda[p]) \subseteq \sum_{\mu \in M} G_\mu$. Taking Λ to be the initial segment of ordinal numbers less than Ω where Ω is the least ordinal of cardinality $|\Lambda|$ and using an argument similar to that employed in [6], we easily obtain an ascending chain

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_\lambda \subseteq \cdots, \quad \lambda \in \Lambda,$$

of subsets of Λ such that $I_0 = \emptyset$, $\lambda \in I_{\lambda+1}$, $I_\lambda = \bigcup_{\mu < \lambda} I_\mu$ if λ is a limit ordinal, $|I_\lambda| \leq \aleph_0 |\lambda|$, and $\phi(\sum_{I_\lambda} G_\mu[p]) \subseteq \sum_{I_\lambda} G_\mu$ for each $\lambda \in \Lambda$.

Suppose that $\sum_{I_\lambda} G_\mu = H + K$ where $\phi(\sum_{I_\lambda} G_\mu[p]) = H[p]$. Since $|I_\lambda| \leq \aleph_0 |\lambda| < |\Lambda|$, the induction hypothesis implies that $K = \sum_{I_\lambda} K_\mu$ where K_μ is closed. Now

$$\sum_{I_{\lambda+1}} G_\mu = H + \left(\sum_{I_\lambda} K_\mu + \sum_{I_{\lambda+1}-I_\lambda} G_\mu \right).$$

Let π be the projection of $\sum_{I_{\lambda+1}} G_\mu$ onto the second summand, $\sum K_\mu + \sum G_\mu$. Set $T = \pi\phi((\sum K_\mu + \sum G_\mu)[p])$. Then

$$\phi\left(\sum_{I_{\lambda+1}} G_\mu[p]\right) = H[p] + T.$$

Furthermore, $\pi\phi$ is a projection of $(\sum_{I_\lambda} K_\mu + \sum_{I_{\lambda+1}-I_\lambda} G_\mu)[p]$ that does not decrease heights; once again by the induction hypothesis, T supports a direct summand L_λ of $\sum_{I_\lambda} K_\mu + \sum_{I_{\lambda+1}-I_\lambda} G_\mu$ which is a direct sum of no more than $\aleph_0 |I_{\lambda+1}|$ closed groups. It follows that $L = \sum_{\lambda \in \Lambda} L_\lambda$ is a pure subgroup of G supported by S . A close analysis of the proof of Theorem 1 reveals that L is a direct summand of G . Since any direct summand of G supported by S is isomorphic to L , the theorem is proved.

COROLLARY 3. *A direct summand of a direct sum of closed groups is necessarily a direct sum of closed groups.*

The theorem by Enochs cited earlier and Corollary 3 prove

THEOREM 4. *If G is a direct sum of closed groups, then any two direct decompositions of G have isomorphic refinements.*

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