A NOTE ON CARTAN'S THEOREMS A AND B

YUM-TONG SIU

In this short note we show that Cartan's Theorem A is an easy consequence of Cartan's Theorem B (for both reduced and unreduced complex spaces).

Assume that Theorem B is true. Suppose $\mathfrak{F}$ is a coherent analytic sheaf on a Stein space $(X, \mathcal{O})$ and $x \in X$. Let $\mathfrak{g}$ be the sheaf of germs of holomorphic functions vanishing at x. We have an epimorphism $\phi: \mathcal{O}_x^p \to \mathfrak{F}_x$. Let $e_k = (0, \cdots, 0, 1, 0, \cdots, 0) \in \mathcal{O}_x^p$, (the 1 is in the kth position, $1 \leqq k \leqq p$). Consider the exact sequence of sheaf-homomorphisms

$$(*) \qquad 0 \longrightarrow \mathfrak{g}\mathfrak{F} \longrightarrow \mathfrak{F} \xrightarrow{\psi} \mathfrak{F}/\mathfrak{g}\mathfrak{F} \longrightarrow 0.$$

Since $(\mathfrak{F}/\mathfrak{g}\mathfrak{F})_y = 0$ for $y \neq x$, $(\psi \circ \phi)(e_k) \in (\mathfrak{F}/\mathfrak{g}\mathfrak{F})_x$ define sections $s_k \in \Gamma(X, \mathfrak{F}/\mathfrak{g}\mathfrak{F})$, $1 \leqq k \leqq p$. Cartan's Theorem B implies that $H^1(X, \mathfrak{g}\mathfrak{F}) = 0$. From the exact cohomology sequence of (*) we conclude that ψ induces an epimorphism $\Gamma(X, \mathfrak{F}) \to \Gamma(X, \mathfrak{F}/\mathfrak{g}\mathfrak{F})$. There exist $f_k \in \Gamma(X, \mathfrak{F})$ such that $\psi(f_k) = s_k$, $1 \leqq k \leqq p$. We claim that $(f_1)_x, \cdots, (f_p)_x$ generate $\mathfrak{F}_x$. Take $u \in \mathfrak{F}_x$. Then $\phi(v) = u$ for some $v \in \mathcal{O}_x^p$. $\psi(\phi(e_k) - (f_k)_x) = 0$ implies that $\phi(e_k) - (f_k)_x \in \mathfrak{g}_x \mathfrak{F}_x = \mathfrak{g}_x \phi(\mathcal{O}_x^p) = \phi(\mathfrak{g}_x \mathcal{O}_x^p)$. $\phi(e_k) - (f_k)_x = \phi(g_k)$ for some $g_k = (g_{k1}, \cdots, g_{kp}) \in \mathfrak{g}_x \mathcal{O}_x^p$, $1 \leqq k \leqq p$. $\det(\delta_{kl} - g_{kl})_{1 \leqq k \leqq p, 1 \leqq l \leqq p}$ is a unit in $\mathcal{O}_x$, where $\delta_{kl} = 1 \in \mathcal{O}_x$ if $k = l$, and $\delta_{kl} = 0 \in \mathcal{O}_x$ if $k \neq l$. $e_1 - g_1, \cdots, e_p - g_p$ generate $\mathcal{O}_x^p$. $v = \sum_{k=1}^p \lambda_k (e_k - g_k)$ for some $\lambda_k \in \mathcal{O}_x$, $1 \leqq k \leqq p$. $u = \phi(v) = \sum_{k=1}^p \lambda_k \phi(e_k - g_k) = \sum_{k=1}^p \lambda_k (f_k)_x$. Theorem A is proved. Moreover, we have proved that the least number of global sections required to generate a given stalk is the same as the least number of elements of the stalk to generate the stalk. This gives us the following:

COROLLARY. *Suppose Z is a subvariety in an open subset $(G, \mathcal{H})$ of a complex number space. The set of points $\sigma(Z)$ of Z, at which Z is not an algebraic complete intersection, is a subvariety.*

PROOF. Without loss of generality we can assume that G is Stein. Let the ideal-sheaf of Z be $\mathfrak{a}$. If Z is of pure codimension p, then

$$\sigma(Z) = \bigcap \left\{ \left\{ z \,\middle|\, z \in Z, \left(\sum_{i=1}^{p} \mathcal{H} f_i \right)_z \neq \mathfrak{a}_z \right\} \,\middle|\, f_1, \cdots, f_p \in \Gamma(G, \mathfrak{a}) \right\}$$

Received by the editors March 13, 1967.

and hence is a subvariety. If $Z=\bigcup_q Z^q$ is the decomposition into pure dimensional subvarieties, where codim $Z^q=q$, then since at points not in $\sigma(Z)$, Z is pure dimensional, $\sigma(Z)=(\bigcup_q \sigma(Z^q))\cup(\bigcup_{q\neq r}(Z^q\cap Z^r))$. Q.E.D.

REFERENCE

1. R. C. Gunning and H. Rossi, *Analytic functions of several complex variables* Prentice-Hall, Englewood Cliffs, N. J., 1965.

PURDUE UNIVERSITY

ON THE MAGNITUDE OF x^n-1 IN A NORMED ALGEBRA

LAWRENCE J. WALLEN

In their work on topological near-rings, Beidleman and Cox[1] had occasion to prove the following theorem: If A is a linear transformation on C^n (with a norm $\|\cdot\|$) and if $\|A^n-I\|\leqq\alpha$ for all n with $0\leqq\alpha<1$, then $A=I$. Their proof consists in noting that $A-I$ is nilpotent and then examining the Jordan form.

It is certainly natural to inquire whether this theorem extends to the case where A is a bounded linear transformation on a normed linear space, or is in a normed algebra. Below we give an extremely short proof that the theorem does extend.

THEOREM. *If X is a normed algebra with* 1 *and $\alpha_n=\|x^n-1\|$ satisfies $\alpha_n=o(n)$ and* $\lim\inf n^{-1}(\alpha_1+\alpha_2+\cdots+\alpha_{n-1})=\beta<1$, *then $x=1$. In particular $x=1$ if $\alpha_n\leqq\alpha<1$.*

PROOF. Write

$$(x-1)=\frac{x^n-1}{n}+(x-1)\left[1-\frac{1+x+\cdots+x^{n-1}}{n}\right]$$

$$=\frac{(x^n-1)}{n}+\frac{(x-1)}{n}\left[(1-x)+(1-x^2)+\cdots+(1-x^{n-1})\right].$$

Hence $\|x-1\|\leqq[\alpha_n+\|x-1\|(\alpha_1+\alpha_2+\cdots+\alpha_{n-1})]/n$. Letting n tend to ∞ through a suitable sequence, we get $\|x-1\|\leqq\beta\|x-1\|$ giving the result.

STEVENS INSTITUTE OF TECHNOLOGY

Received by the editors March 23, 1967.

[1] Private communication.