TAUBERIAN THEOREMS FOR ABSOLUTE SUMMABILITY

J. S. RATTI

1.1. Let $\sum a_n$ be an infinite series, and let $\{\lambda_n\}$ be an arbitrary sequence of positive numbers tending to infinity with n, such that

$$1 \leqq \lambda_1 < \lambda_2 < \cdots.$$

We write

$$A_\lambda^k(x) = \sum_{\lambda_n < x} (x - \lambda_n)^k a_n = \int_1^x (x - t)^k dA_\lambda(t),$$

$$A_\lambda^0(x) = A_\lambda(x) = \sum_{\lambda_n < x} a_n,$$

$$A_\lambda^k(x) = 0 \qquad \text{for } x \leqq 1 \quad \text{and} \quad k > -1.$$

We also write

$$B_\lambda^k(x) = \sum_{\lambda_n < x} (x - \lambda_n)^k \lambda_n a_n.$$

The series $\sum a_n$ is said to be summable $|R, \lambda, k|_m$, $k>0$, $m \geqq 1$, $k > 1 - 1/m$, if

$$\int_1^\infty x^{m-1} \left| \frac{d}{dx} x^{-k} A_\lambda^k(x) \right|^m dx < \infty.$$

The case $\lambda_n = n$ of this definition is given in [1] where it is shown why the additional restriction $k > 1 - 1/m$ is necessary.

The series $\sum a_n$ is said to be summable $|A, \lambda|_m$, $m \geqq 1$, if the series $f(x) = \sum a_n \exp[-\lambda_n x]$ converges for $x > 0$ and

$$\int_0^\infty (1 - e^{-x})^{m-1} | f'(x) |^m dx < \infty,$$

[8, Theorem 2].

It is easily seen that for $m = 1$, summability $|A, \lambda|_m$ and summability $|R, \lambda, k|_m$ are the same as summability $|A, \lambda|$ [9] and summability $|R, \lambda, k|$ [2] respectively. Borwein [1] has shown that for $\lambda_n = n$ summability $|R, \lambda, k|_m$ of $\sum a_n$ is equivalent to its absolute Cesàro summability with index m.

1.2. Hyslop [6] has established the following Tauberian theorem for absolute summability.

Received by the editors August 22, 1966.

THEOREM A. *If* $\sum a_n$ *is summable* $|A|$ *and* $\sum \Delta(na_n)$ *is summable* $|C, k+1|$, *where* $k \geqq 0$, *then* $\sum a_n$ *is summable* $|C, k|$.

Flett [3] generalized Theorem A for index m and proved the following theorem.

THEOREM B. *Let* $m \geqq 1$, $k > -1$. *If* $\sum a_n$ *is summable* $|A|_m$ *and if also* $\sum \Delta(na_n)$ *is summable* $|C, k+1|_m$, *then* $\sum a_n$ *is summable* $|C, k|_m$.

The object of this paper is to obtain an analogue of Theorem B involving the extended definitions of absolute Abel and absolute Riesz summability referred to above as summability $|A, \lambda|_m$ and summability $|R, \lambda, k|_m$.

I wish to express my appreciation to Professor D. Waterman for his guidance during the preparation of this paper. I also thank the referee for his valuable suggestions, especially Theorem 3.

2.1. We prove the following theorems.

THEOREM 1. *If* (i) $\sum a_n$ *is summable* $|A, \lambda|_m$, $m \geqq 1$, *then a necessary and sufficient condition for the given series* $\sum a_n$ *to be summable* $|R, \lambda, k|_m$ *is that*

(ii) $\int_0^\infty x^{m-1} |(d/dx) x^{-k-1} B_\lambda^k(x)|^m dx < \infty$, *where* $k > 1 - 1/m$.

THEOREM 2. *If* (i) $x^{-1}B_\lambda^0(x)$ *is of bounded variation in* $(0, \infty)$ *and* (ii) *the series* $\sum a_n \exp[-\lambda_n x]$ *is convergent for all* $x > 0$ *and its sum* $f(x)$ *is of bounded variation in* $(0, \infty)$, *then* $\sum a_n$ *is absolutely convergent. Moreover, the conditions* (i) *and* (ii) *are also necessary for the absolute convergence of* $\sum a_n$.

2.2. We require the following lemmas for the proof of our theorem.

LEMMA 1 [5]. *If* $k > 0$, *then*

$$d/dx(x^{-k} A_\lambda^k(x)) = kx^{-k-1} B_\lambda^{k-1}(x).$$

LEMMA 2 [2]. *If* $f(x) = \sum a_n \exp[-\lambda_n x]$ *converges for* $x > 0$, *then for* $k \geqq 0$ *we have*

$$f(x) = \frac{x^{k+1}}{\Gamma(k+1)} \int_0^\infty A^k(t) e^{-xt} dt.$$

LEMMA 3. *For* (i) $\sum a_n$ *to be summable* $|R, \lambda, k|_m$ *it is necessary and sufficient that*

(ii) $\sum a_n$ *be summable* $|R, \lambda, k+1|_m$ *and*

(iii) $\int_0^\infty x^{m-1} |(d/dx) x^{-k-1} B_\lambda^k(x)|^m dx < \infty$, *where* $m \geqq 1$, $k > 1 - 1/m$.

PROOF OF LEMMA 3. (a) Let $m > 1$. Since $A_\lambda^k(x) = 0$ for $x \leqq 1$, we write

$$\int_0^\infty x^{m-1}\left|\frac{d}{dx}x^{-k}A_\lambda^k(x)\right|^m dx = \int_1^\infty x^{m-1}\left|\frac{d}{dx}x^{-k}A_\lambda^k(x)\right|^m dx.$$

We have

$$\frac{d}{dx}(x^{-k-1}A_\lambda^{k+1}(x)) = (k+1)x^{-k-1}A_\lambda^k(x) - (k+1)x^{-k-2}A_\lambda^{k+1}(x).$$

Using Lemma 1, we get

$$x^{-k}A_\lambda^k(x) = x^{-k-1}A_\lambda^{k+1}(x) + x^{-k-1}B_\lambda^k(x). \tag{2.2.1}$$

Therefore

$$\int_0^\infty x^{m-1}\left|\frac{d}{dx}x^{-k}A_\lambda^k(x)\right|^m dx \leqq M\int_0^\infty x^{m-1}\left|\frac{d}{dx}x^{-k-1}A_\lambda^{k+1}(x)\right|^m dx + M\int_0^\infty x^{m-1}\left|\frac{d}{dx}x^{-k-1}B_\lambda^k(x)\right|^m dx.$$

The sufficiency follows, since by (ii) and (iii) the right-hand side is finite. Throughout this paper M denotes a positive constant which is not necessarily the same at every occurrence.

Since summability $|R, \lambda, k|_m$ implies summability $|R, \lambda, k+1|_m$ [7], using (2.2.1) we get

$$\int_0^\infty x^{m-1}\left|\frac{d}{dx}x^{-k-1}B_\lambda^k(x)\right|^m dx \leqq M\int_0^\infty x^{m-1}\left|\frac{d}{dx}x^{-k}A_\lambda^k(x)\right|^m dx + M\int_0^\infty x^{m-1}\left|\frac{d}{dx}x^{-k-1}A_\lambda^{k+1}(x)\right|^m dx.$$

The right-hand side is again finite, the lemma follows.

(b) For $m=1$, the lemma follows at once from (2.2.1) and the first theorem of consistency [2] for $|R, \lambda, k|$ summability of $\sum a_n$.

REMARK. It should be pointed out that Lemma 3 continues to hold when $m=1$ and $k=0$. It is easy to see that (2.2.1) holds for $k=0$. Since

$$\sum_{\lambda_n<x} a_n - \sum_{\lambda_n<x}\left(1-\frac{\lambda_n}{x}\right)a_n = x^{-1}B_\lambda^0(x). \tag{2.2.2}$$

2.3. PROOF OF THEOREM 1. The necessity part of the theorem follows from Lemma 3. Therefore we have only to prove that conditions (i) and (ii) are sufficient for the summability $|R, \lambda, k|_m$ of the given series. In view of Lemma 3, it is sufficient that the given series be summable $|R, \lambda, k+1|_m$, i.e.

$$I = \int_0^\infty x^{m-1} \left| (k+1) x^{-k-2} B_\lambda^k(x) \right|^m dx < \infty.$$

By Lemma 2, on account of hypothesis (i), we have

$$f'(x) = -\frac{x^{k+1}}{(k+1)} \int_0^\infty B_\lambda^k(t) e^{-xt} dt.$$

Again by hypothesis (i)

$$\int_0^\infty (1 - e^{-x})^{m-1} \left| f'(x) \right|^m dx < \infty,$$

and by second mean value theorem, for some R, $0 < R < \infty$, this integral is equal to

$$\int_0^R (e^x - 1)^{m-1} \left| f'(x) \right|^m dx$$

$$= \int_{1/R}^\infty (e^{1/x} - 1)^{m-1} \left| f'\left(\frac{1}{x}\right) \right|^m x^{-2} dx > \int_{1/R}^\infty x^{-m-1} \left| f'\left(\frac{1}{x}\right) \right|^m dx.$$

Hence it will suffice to show that

$$J = \int_{1/R}^\infty \left| (k+1) x^{-k-1-1/m} B_\lambda^k(x) - \frac{x^{-k-2-1/m}}{\Gamma(k+1)} \int_0^\infty B_\lambda^k(t) e^{-t/x} dt \right|^m dx < \infty.$$

Writing $g(x)$ for $x^{-k-1} B_\lambda^k(x)$, we have

$$J \leqq M \int_0^\infty x^{m-1} \left| g(x) x^{-k-3} \frac{1}{\Gamma(k+2)} \int_0^\infty t^{k+1} e^{-t/x} dt \right.$$

$$\left. - \frac{x^{-k-3}}{\Gamma(k+2)} \int_0^\infty B_\lambda^k(t) e^{-t/x} dt \right|^m dx,$$

since

$$\frac{x^{-k-2}}{\Gamma(k+2)} \int_0^\infty t^{k+1} e^{-t/x} dt = 1.$$

Now

$$J \leqq M \int_0^\infty x^{-mk-2m-1} dx \left| \int_0^x \{ g(x) - g(t) \} t^{k+1} e^{t/x} dt \right.$$

$$\left. - \int_x^\infty \{ g(t) - g(x) \} t^{k+1} e^{-t/x} dt \right|^m.$$

By Minkowski's inequality;

$$J^{1/m} \leqq J_1^{1/m} + J_2^{1/m},$$

where

$$J_1 = M\left\{\int_0^\infty x^{-mk-2m-1}\left|\int_0^x t^{k+1}e^{-t/x}dt\int_t^x dg(u)\right|^m dx\right\}$$

and

$$J_2 = M\left\{\int_0^\infty x^{-mk-2m-1}\left|\int_x^\infty t^{k+1}e^{-t/x}dt\int_x^t dg(u)\right|^m dx\right\}.$$

We will show that J_1 is finite. Putting $t=uy$ in the t-integral of J_1, we get

$$J_1 \leqq M\int_0^\infty x^{-mk-2m-1}dx\left\{\int_0^x u^{k+2}\,|\,dg(u)\,|\int_0^1 y^{k+1}e^{-uy/x}dy\right\}^m$$

$$\leqq M\int_0^\infty x^{-mk-m-1}dx\left\{\int_0^x u^{k+1}\,|\,dg(u)\,|\right\}^m.$$

Applying Hölder's inequality to the u-integral, we get

$$J_1 \leqq M\int_0^\infty x^{-mk-2}dx\int_0^x u^{mk+m}\,|\,dg(u)\,|^m,$$

$$= M\int_0^\infty u^{mk+m}\,|\,dg(u)\,|^m\int_u^\infty x^{-mk-2}dx$$

$$= M\int_0^\infty u^{m-1}\,|\,dg(u)\,|^m$$

$$< \infty,$$

by hypothesis (ii). We will now show that J_2 is finite.

Case (a). Let $m>1$. Putting $t=uy$ in the t-integral of J_2, we get

$$J_2 \leqq M\int_0^\infty x^{-mk-2m-1}dx\left\{\int_x^\infty u^{k+2}\,|\,dg(u)\,|\int_1^\infty y^{k+1}e^{-uy/x}dy\right\}^m.$$

Since

$$\int_1^\infty y^{k+1}e^{-uy/x}dy < e^{1-u/x}\int_1^\infty y^{k+1}e^{-y}dy = Me^{-u/x},$$

we have

$$J_2 \leqq M\int_0^\infty x^{-mk-2m-1}dx\left\{\int_x^\infty u^{k+2}e^{-u/x}\,|\,dg(u)\,|\right\}^m.$$

Applying Hölder's inequality, we obtain with $1/m+1/m'=1$,

$$J_2 \leqq M\int_0^\infty x^{-mk-2m-1}dx\left\{\int_x^\infty |dg(u)|^m\right\}\left\{\int_x^\infty u^{km'+2m'}e^{-m'u/x}du\right\}^{m-1}$$

$$\leqq M\int_0^\infty x^{-mk-2m-1}dx\left\{\int_x^\infty |dg(u)|^m\right\}\left\{\int_0^\infty u^{km'+2m'}e^{-m'u/x}du\right\}^{m-1}$$

$$= M\int_0^\infty x^{m-2}dx\int_x^\infty |dg(u)|^m$$

$$= M\int_0^\infty |dg(u)|^m\int_0^u x^{m-2}dx$$

$$= M\int_0^\infty u^{m-1}|dg(u)|^m$$

$$< \infty,$$

by hypothesis (ii).

Case (b). Let $m=1$. We have

$$J_2 \leqq M\int_0^\infty x^{-k-3}dx\int_x^\infty u^{k+2}e^{-u/x}|dg(u)|$$

$$= M\int_0^\infty u^{k+2}|dg(u)|\int_0^u x^{-k-3}e^{-u/x}dx$$

$$= M\int_0^\infty |dg(u)|\int_1^\infty v^{k+1}e^{-v}dv$$

$$= M\int_0^\infty |dg(u)|$$

$$< \infty,$$

by hypothesis (ii). This completes the proof of Theorem 1.

Proof of Theorem 2. The first part of the theorem follows from the sufficiency part of Theorem 1 with $k=0$. The necessity of condition (i) follows from the remark following Lemma 3. The necessity of condition (ii) is obvious.

In the following theorem, we prove the Abelian relationship between the summability $|R, \lambda, k|_m$ and the summability $|A, \lambda|_m$ of $\sum a_n$.

Theorem 3. *If* (i) $\sum a_n$ *is summable* $|R, \lambda, k|_m$, $m\geqq 1$, $k>1-1/m$, *and* (ii) *the series* $\sum a_n \exp[-\lambda_n x]$ *converges for all* $x>0$ *to the sum* $f(x)$, *then* (iii) $\sum a_n$ *is summable* $|A, \lambda|_m$.

PROOF. In virtue of (ii),

$$f'(x) = -\frac{x^{k+1}}{\Gamma(k+1)}\int_0^\infty B_\lambda^k(t)e^{-xt}dt.$$

Thus

$$\int_0^\infty (1-e^{-x})^{m-1}|f'(x)|^m dx$$

$$\leqq M\int_0^\infty (1-e^{-x})^{m-1}x^{mk+m}dx\left\{\int_0^\infty |B_\lambda^k(t)|^m e^{-xt}dt\right\}\left\{\int_0^\infty e^{-xt}dt\right\}^{m-1}$$

$$= M\int_0^\infty |B_\lambda^k(t)|^m dt\int_0^\infty \left(\frac{1-e^{-x}}{x}\right)^{m-1}x^{mk+m}e^{-xt}dt$$

$$\leqq M\int_0^\infty t^{-m-mk-1}|B_\lambda^k(t)|^m dt$$

$$< \infty,$$

by Lemmas 1 and 3.

In the special case $\lambda_n = n$, it follows from results of Borwein [1] and Flett [4] that (i) implies (iii); so that, in this case, (i) alone implies (iii), but this is probably not true for more general λ_n.

REFERENCES

1. D. Borwein, *An extension of a theorem on the equivalence between absolute Rieszian and absolute Cesàro summability*, Proc. Glasgow Math. Assoc. **4** (1959) 81–83.

2. K. Chandrasekhran and S. Minakshisundram, *Typical means*, Oxford Univ. Press, Oxford, 1952.

3. T. M. Flett, *Some generalizations of Tauber's second theorem*, Quart. J. Math. Oxford Ser. II **10** (1959), 70–80.

4. ———, *On an extension of absolute summability and some theorems of Littlewood and Paley*, Proc. London Math. Soc. (3) **7** (1957), 113–141.

5. J. M. Hyslop, *On the absolute summability of series by Rieszian means*, Proc. Edinburgh Math. Soc. (2) **5** (1936), 46–54.

6. ———, *A Tauberian theorem for absolute summability*, J. London Math. Soc. **12** (1937), 176–180.

7. J. S. Ratti, *On high indices theorems*, Proc. Amer. Math. Soc. **17** (1966), 1001–1006.

8. D. Waterman, *On some high indices theorems*, Trans. Amer. Math. Soc. **69** (1950), 468–478.

9. A. Zygmund, *On certain integrals*, Trans. Amer. Math. Soc. **55** (1944), 170–204.

WAYNE STATE UNIVERSITY AND
OAKLAND UNIVERSITY