INEQUALITIES FOR THE DIFFERENCE BODY OF A CONVEX BODY

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1. Introduction. Let K be a convex body, i.e. a compact convex set having interior points, in n -dimensional Euclidean space E_n . The "difference body" of K , denoted by DK , is the centrally symmetric convex body (centered at the origin) defined by,

$$(1) \quad DK = K + (-K) = \{x - y: x \in K, y \in K\}.$$

It is well known that DK can equivalently be described as follows,

$$(2) \quad DK = \{x: (x + K) \cap K \neq \emptyset\}.$$

In the following, S will denote the boundary of the unit ball in E_n , and u a variable point of S (so u is a unit vector, or "direction"). The polar equation of the boundary of DK is given by $\rho = \rho(u)$, $u \in S$, so $\rho(u)$ is the radius of DK in the direction u . Then $\rho(u)$ is the maximum length of a chord of K having direction u —the length of a "diameter" of K having direction u .

Let μ denote n -dimensional Lebesgue measure in E_n . Then Rogers and Shephard [5] proved that

$$(3) \quad \mu(DK) \leq \binom{2n}{n} \mu(K),$$

where equality holds if and only if K is a simplex. They later gave a more geometrical proof in [6]. In §2 we give a proof of (3) which clarifies its relationship with the Brunn-Minkowski Theorem and leads to the following generalization, proved at the end of §3. Let $\alpha \geq 1 - n$, and let $B(p, q)$ be the Beta function, defined by

$$(4) \quad B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt.$$

Then

$$(5) \quad \int_{K \times K} |x - y|^\alpha d\mu(x) d\mu(y) \geq B(n + \alpha, n + 1) \mu(K) \int_S \rho^{n+\alpha}(u) du,$$

where $|x - y|$ is the distance, in E_n , between x and y . Equality holds in (5) if and only if K is a simplex. One sees that (3) follows from (5) by setting $\alpha = 0$.

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2. The proof of (3) given in [5] depended on a lemma of T. Bang [1, Lemma I]. One can derive that lemma from the following, proved by Fáy and Rédei [4].

LEMMA 1. *Let K_1 and K_2 be convex bodies in E_n . Let $\mathfrak{F}$ be the family of convex sets $(x + K_1) \cap K_2$, with x ranging over $B = \{x: (x + K_1) \cap K_2 \neq \emptyset\}$. Then $\mathfrak{F}$ is a concave family.*

The Brunn-Minkowski Theorem then implies the following lemma, which is simply a restatement of [4, Satz 3].

LEMMA 2. *Let K_1 and K_2 be convex bodies in E_n , and let f be the function defined by $f(x) = \mu((x + K_1) \cap K_2)^{1/n}$, with domain $B = \{x: (x + K_1) \cap K_2 \neq \emptyset\}$. Then f is a concave function on B .*

The following proof of (3) is only a slight variant of the first proof of Rogers and Shephard.

PROOF OF (3). Let (r, u) be spherical coordinates for E_n , so r represents distance from 0 and u ranges over the unit sphere S centered at 0. Let $f(x) = \mu((x + K) \cap K)^{1/n}$, for $x \in DK$. Then f is concave, by Lemma 2, and $(f(0))^n = \mu(K)$. Let $g(x) = f(0)(1 - r/\rho)$, $x \in DK$, where $r = |x|$ is the distance from x to 0, and $\rho = \rho(x/r)$ is the radius of DK in the direction of x . Then g vanishes on the boundary of DK , $g(0) = f(0)$, and g is linear on each segment joining 0 to the boundary of DK (the ordinate set of g is a cone in E_{n+1} with altitude $f(0)$, base DK , and is contained in the ordinate set of f). The concavity of f implies $f(x) \geq g(x)$, $x \in DK$. Thus

$$\begin{aligned}
 \int_{DK} \mu((x + K) \cap K) d\mu(x) &= \int_{DK} f^n d\mu \geq \int_{DK} g^n d\mu \\
 &= (f(0))^n \int_S \int_0^\rho (1 - r/\rho)^n r^{n-1} dr du \\
 (6) \quad &= \mu(K) \int_S \rho^n \int_0^1 t^{n-1} (1 - t)^n dt du \\
 &= \binom{2n}{n}^{-1} \mu(K) \frac{1}{n} \int_S \rho^n du = \binom{2n}{n}^{-1} \mu(K) \mu(DK).
 \end{aligned}$$

With the following easy identity (see [5, (7)]),

$$(7) \quad \mu(K)^2 = \int_{DK} \mu((x + K) \cap K) d\mu(x),$$

the inequality (3) follows. Equality holds in (3) if and only if equality holds throughout in (6), so that $f = g$. Then f is linear on each segment

joining 0 to the boundary of DK . By the Brunn-Minkowski Theorem, this happens if and only if $(x+K) \cap K$ is positively homothetic to $K = (0+K) \cap K$ whenever $(x+K) \cap K$ is nondegenerate. It is proved in [5] that this property characterizes simplices; hence equality holds in (3) if and only if K is a simplex. (The proof of this characterization of simplices is simplified in [3].) This completes the proof.

The idea used in the above proof suggests the following theorem, whose proof is omitted since it follows precisely the same lines as above.

THEOREM 1. *Let f be a nonnegative concave function defined on a convex body B in E_n . Let $x_0 \in B$, and let $h(\xi)$, $\xi \geq 0$, be a strictly increasing real-valued function of a real variable. Then*

$$(8) \quad \int_B h(f(x)) d\mu(x) \geq n\mu(B) \int_0^1 h(tf(x_0))(1-t)^{n-1} dt,$$

where equality holds if and only if f vanishes on the boundary of B and is linear on each segment joining x_0 to the boundary of B .

Note that (3) follows from (8) by taking $B = DK$, $x_0 = 0$, $f(x) = \mu((x+K) \cap K)^{1/n}$, $h(\xi) = \xi^n$.

The following theorem, which was used in [6] in order to prove (3), follows immediately from Theorem 1.

THEOREM 2. *Let K be an $(r+s)$ -dimensional convex body. Let R be an r -dimensional section of K with r -measure $\mu(R)$, and let S be the projection of K onto an s -dimensional subspace totally orthogonal to R . Let $\mu(S)$ be the s -measure of S . Then*

$$(9) \quad \binom{r+s}{s} \mu(K) \geq \mu(R)\mu(S),$$

where equality holds if and only if each r -dimensional section of K parallel to R is positively homothetic to R .

PROOF. In Theorem 1, take $n = s$, $B = S$, $h(\xi) = \xi^r$, $f(x) = \mu(R(x))^{1/r}$, for $x \in S$, where $R(x)$ is the r -dimensional section of K by the r -flat orthogonal to S passing through x , and $\mu(R(x))$ is the r -measure of $R(x)$. Let $R(x_0) = R$. By the Brunn-Minkowski Theorem, f is concave on S . Hence, from (8), we have,

$$(10) \quad \begin{aligned} \mu(K) &= \int_S \mu(R(x)) d\mu(x) \geq s\mu(S)\mu(R) \int_0^1 t^r(1-t)^{s-1} dt \\ &= \mu(S)\mu(R) \binom{r+s}{s}^{-1}. \end{aligned}$$

Equality holds if and only if f is linear on each segment joining x_0 to the boundary of S . This is the case if and only if all the $R(x)$ are homothetic. This completes the proof.

3. The position of an oriented line G in E_n can be fixed by specifying its direction u and the point p where G intersects the orthogonal hyperplane through the origin. The "integral-geometric density" dG , for oriented lines is then given by

$$(11) \quad dG = dp du,$$

where dp is the $(n-1)$ -dimensional volume element in the orthogonal hyperplane, and du is the element of surface area of the unit sphere S . If K is a convex body in E_n , let $\sigma(G) = \sigma(p, u)$ be the length of the chord $K \cap G$, for each oriented line $G = G(p, u)$, where p and u are as described above. Let

$$(12) \quad I_\alpha(K) = \int \sigma^\alpha(G) dG = \int_S \int_{K_u} \sigma^\alpha dp du,$$

where K_u is the orthogonal projection of K onto a hyperplane orthogonal to u , and dp is the volume element in that hyperplane.

THEOREM 3. *Let K be a convex body in E_n , and let $\rho = \rho(u)$, $u \in S$, be the polar equation of DK . Then if m is a real number ≥ 1 , we have*

$$(13) \quad I_{m+1}(K) \geq m(m+1)B(m, n+1)\mu(K) \int_S \rho^m(u) du,$$

where equality holds if and only if K is a simplex.

PROOF. The concavity of $\mu((x+K) \cap K)^{1/n}$ implies (see the proof of (3) in §2)

$$(14) \quad \mu((x+K) \cap K) \geq \mu(K)(1 - r/\rho)^n,$$

with $r = |x|$, and $\rho = \rho(x/r) = \rho(u)$, $x \in DK$. Multiply both sides of (14) by r^{m-1} and integrate with respect to r , to obtain

$$(15) \quad \int_0^\rho r^{m-1} \mu((x+K) \cap K) dr \geq \mu(K) \int_0^\rho r^{m-1} (1 - r/\rho)^n dr \\ = \mu(K) B(m, n+1) \rho^m(u).$$

Let $l(p, u)$ be the length of the intersection of the line $G(p, u)$ with $(x+K) \cap K$, where $u = x/|x|$. Then $l(p, u) = \sigma(p, u) - r$, when $\sigma \geq r$, and

$$\begin{aligned}
 \int_0^{\rho} r^{m-1} \mu((x+K) \cap K) dr &= \int_0^{\rho} r^{m-1} \int_{K_u} l(p, u) dp dr \\
 &= \int_{K_u} \int_0^{\rho} r^{m-1} l(p, u) dr dp \\
 &= \int_{K_u} \int_0^{\sigma} r^{m-1} (\sigma - r) dr dp \\
 &= \frac{1}{m(m+1)} \int_{K_u} \sigma^{m+1}(p, u) dp.
 \end{aligned}
 \tag{16}$$

Then (15) and (16) yield

$$\frac{1}{m(m+1)} \int_{K_u} \sigma^{m+1}(p, u) dp \geq \mu(K) B(m, n+1) \rho^m(u).
 \tag{17}$$

Integration of (17) with respect to u yields (13). Equality holds in (13) if and only if it holds in (14), for all $x \in DK$. It follows (as in the proof of (3) in §2) that this happens if and only if K is a simplex. This completes the proof.

In (16), set $m=n$ and integrate both sides over S . This gives

$$\int_S \int_0^{\rho} \mu((x+K) \cap K) r^{n-1} dr du = \frac{1}{n(n+1)} I_{n+1}(K).
 \tag{18}$$

Using (7), we then have

$$I_{n+1}(K) = n(n+1) \mu(K)^2.
 \tag{19}$$

This is a generalization to E_n of an integral-geometric formula of Crofton (see [2, p. 20]). If we set $m=n$ in (13) and use (19), we obtain the Rogers-Shephard inequality (3).

In order to prove (5), we need a generalization of the formula of Crofton given in [2, p. 19]. The integral-geometric density for pairs of points $x, y \in E_n$ is $d\mu(x)d\mu(y)$. If $G=G(p, u)$ is the line through x and y , then a straightforward computation shows that

$$d\mu(x)d\mu(y) = |x-y|^{n-1} dt_1 dt_2 dp du,
 \tag{20}$$

where $|x-y|$ is the distance from x to y , t_1 is the distance of x from a fixed point of G , and t_2 is the distance of y from a fixed point of G . Multiplying both sides of (20) by $|x-y|^\alpha$, $\alpha \geq 1-n$, and integrating over all pairs of points $x, y \in K$ yields

$$(21) \quad \int_{K \times K} |x - y|^{\alpha} d\mu(x) d\mu(y) = \frac{1}{(n + \alpha)(n + \alpha + 1)} I_{n+\alpha+1}(K).$$

If we set $m = n + \alpha$ in (13), and apply (21), we obtain (5).

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