AN INEQUALITY FOR LINEAR TRANSFORMATIONS[1]

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1. **Statements of results.** In this paper the following elementary inequality is proved and exploited.

THEOREM 1. *If L is a positive-definite hermitian transformation on the finite dimensional unitary space V and $p \geqq 1$, then for arbitrary vectors u and v*

$$\|u\|^2 + (L^{-p}v, v) \geqq ((I + L)^{-p}u + v, u + v). \tag{1}$$

From (1) we can conclude

THEOREM 2. *If H and K are positive-definite hermitian transformations on V and x and y are arbitrary vectors, then*

$$(H^{-1}x, x) + (K^{-1}y, y) \geqq ((H + K)^{-1}x + y, x + y). \tag{2}$$

By a trivial induction on (2) the following corollary is obtained.

COROLLARY. *If A_k, $k=1, \cdots, m$, are positive-definite hermitian on V and $x_1, \cdots, x_m$ are arbitrary vectors in V, then*

$$\sum_{k=1}^{m} (A_k^{-1}x_k, x_k) \geqq \left(\left(\sum_{k=1}^{m} A_k\right)^{-1} \sum_{k=1}^{m} x_k, \sum_{k=1}^{m} x_k\right). \tag{3}$$

The result (2) implies the following extension of Bergstrom's inequality [1, p. 119].

THEOREM 3. *Let A and B be n-square hermitian matrices and let A_1 be the principal submatrix of A obtained by deleting row one and column one of A. If A_1 and B_1 (defined similarly) are positive-definite hermitian, then*

$$\frac{\det(A + B)}{\det(A_1 + B_1)} \geqq \frac{\det(A)}{\det(A_1)} + \frac{\det(B)}{\det(B_1)} \cdot \tag{4}$$

We also prove

THEOREM 4. *Let H and K be positive-definite hermitian transformations on V with eigenvalues $h_1 \geqq \cdots \geqq h_n$, $k_1 \geqq \cdots \geqq k_n$ respectively. If $H+K$ has eigenvalues $r_1 \geqq \cdots \geqq r_n$ and $m \leqq n/2$, then*

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$$\sum_{j=1}^{m}\left(\frac{1}{h_j}+\frac{1}{k_{n-j+1}}\right) \geqq 2\sum_{j=1}^{m}\frac{1}{r_j}\cdot \tag{5}$$

2. **Proofs.** In proving (1) we establish a substantially more general result.

THEOREM 5. *Let L be a positive-definite hermitian transformation on the unitary space V,* dim $V=n$. *Let f be a scalar valued function defined on* $(0, \infty)$ *satisfying*

$$f(1) = 1, \quad f(x) > 0, \quad f(1 + x) \geqq 1 + f(x). \tag{6}$$

Then for arbitrary vectors u and v

$$\|u\|^2 + (f(L)^{-1}v, v) \geqq (f(I + L)^{-1}u + v, u + v). \tag{7}$$

PROOF. Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of L with a corresponding orthonormal set of eigenvectors $e_1, \cdots, e_n$. Let $\alpha_t=(u, e_t)$, $\beta_t=(v, e_t)$, $t=1, \cdots, n$, and compute that

$$\begin{aligned}
&\|u\|^2 + (f(L)^{-1}v, v) - (f(I + L)^{-1}u + v, u + v)\\
&= \sum_{t=1}^{n}\left\{|\alpha_t|^2 + \frac{1}{f(\lambda_t)}|\beta_t|^2 - \frac{1}{f(1+\lambda_t)}|\alpha_t+\beta_t|^2\right\}\\
&= \sum_{t=1}^{n}\{f(\lambda_t)f(1+\lambda_t)|\alpha_t|^2 + f(1+\lambda_t)|\beta_t|^2\\
&\qquad\qquad - f(\lambda_t)|\alpha_t+\beta_t|^2\}/f(\lambda_t)f(1+\lambda_t)\\
&\geqq \sum_{t=1}^{n}\{f(\lambda_t)(1+f(\lambda_t))|\alpha_t|^2 + (1+f(\lambda_t))|\beta_t|^2\\
&\qquad - f(\lambda_t)(|\alpha_t|^2 + |\beta_t|^2 + 2|\alpha_t||\beta_t|)\}/f(\lambda_t)f(1+\lambda_t)\\
&= \sum_{t=1}^{n}\{(f(\lambda_t))^2|\alpha_t|^2 + |\beta_t|^2 - 2|\alpha_t||\beta_t|f(\lambda_t)\}/f(\lambda_t)f(1+\lambda_t)\\
&= \sum_{t=1}^{n}(f(\lambda_t)|\alpha_t| - |\beta_t|)^2/f(\lambda_t)f(1+\lambda_t)\\
&\geqq 0.
\end{aligned}$$

By setting $f(x)=x^p$, $p\geqq 1$, (1) follows from (7).

To prove Theorem 2 set $u=H^{-1/2}x$, $v=H^{-1/2}y$ so that

$$(H^{-1}x, x) = \|u\|^2, \quad (K^{-1}y, y) = (K^{-1}H^{1/2}v, H^{1/2}v) = (H^{1/2}K^{-1}H^{1/2}v, v).$$

(The positive-definite determination of the square root is invariably used here.) Set $L=H^{-1/2}KH^{-1/2}$ and compute via (1) with $p=1$ that

$$\begin{aligned}((H+K)^{-1}x+y,\, x+y) &= ((H+K)^{-1}H^{1/2}(u+v),\, H^{1/2}(u+v))\\ &= (H^{1/2}(H+K)^{-1}H^{1/2}u+v,\, u+v) = ((H^{-1/2}(H+K)H^{-1/2})^{-1}u+v,\, u+v)\\ &= ((I+L)^{-1}u+v,\, u+v) \leqq \|u\|^2+(L^{-1}v,\, v)\\ &= (H^{-1}x,\, x)+(H^{1/2}K^{-1}H^{1/2}v,\, v) = (H^{-1}x,x)+(K^{-1}y,\, y).\end{aligned}$$

To prove Theorem 3 we derive an elementary identity for the determinant of an n-square matrix. Thus let X be an n-square matrix and suppose $X(1|1)$ denotes the $(n-1)$-square principal submatrix of X obtained by deleting row 1 and column 1 of X. More generally, if $i_1 < \cdots < i_r$, $j_1 < \cdots < j_s$, are integers between 1 and n let $X(i_1, \cdots, i_r | j_1, \cdots, j_s)$ denote the submatrix of X obtained by deleting rows $i_1, \cdots, i_r$ and columns $j_1, \cdots, j_s$ of X. Now

$$\begin{aligned}\det(X) &= \sum_{j=1}^{n} (-1)^{1+j}x_{1j} \det(X(1|j))\\ &= x_{11}\det(X(1|1)) + \sum_{j=2}^{n}(-1)^{1+j}x_{1j}\det(X(1|j)).\end{aligned} \tag{8}$$

Now

$$\det(X(1|j)) = \sum_{k=2}^{n}(-1)^k x_{k1}\det(X(1,k|1,j)). \tag{9}$$

Let $c_{jk} = (-1)^{k+j}\det(X(1,k|1,j))$, $k, j = 2, \cdots, n$, so that the $(n-1)$-square matrix $C = (c_{jk})$ is the adjugate of $X(1|1)$, i.e. $C = \operatorname{adj} X(1|1)$. Then substituting (9) in (8) produces

$$\begin{aligned}\det(X) &= x_{11}\det(X(1|1))\\ &\quad + \sum_{j=2}^{n} x_{1j}(-1)^{1+j}\sum_{k=2}^{n}(-1)^k x_{k1}\det(X(1,k|1,j))\\ &= x_{11}\det(X(1|1)) - \sum_{j,k=2}^{n} x_{1j}x_{k1}c_{jk}.\end{aligned} \tag{10}$$

Thus, if (,) denotes the standard inner product in the space of $(n-1)$-tuples over the complex numbers, (10) reads

$$\det(X) = x_{11}\det(X(1|1)) - (\operatorname{adj} X(1|1)u,\, v)$$

in which $u = (x_{21}, x_{31}, \cdots, x_{n1})$; $v = (\bar{x}_{12}, \bar{x}_{13}, \cdots, \bar{x}_{1n})$. In case $X = A$ is hermitian we know that $u = v$ and we have in the notation of Theorem 3 with $u_A = (a_{21}, a_{31}, \cdots, a_{n1})$

$$\det(A) = a_{11}\det(A_1) - (\operatorname{adj} A_1 u_A,\, u_A). \tag{11}$$

If we assume that A_1 (and B_1) are positive-definite hermitian, then of course adj A_1 is also, $\det(A_1)>0$, adj $A_1=\det(A_1)A_1^{-1}$ and we have from (11) (applied to A, B and $A+B$)

$$\begin{aligned}\det(A)/\det(A_1) &= a_{11} - (A_1^{-1}u_A, u_A),\\ \det(B)/\det(B_1) &= b_{11} - (B_1^{-1}u_B, u_B),\\ \det(A+B)/\det(A_1+B_1) &= a_{11}+b_{11} - ((A_1+B_1)^{-1}u_{A+B}, u_{A+B}).\end{aligned}$$

Now, $u_{A+B}=u_A+u_B$ so that

$$\begin{aligned}&\det(A+B)/\det(A_1+B_1) - \det(A)/\det(A_1) - \det(B)/\det(B_1)\\ &\quad = (A_1^{-1}u_A, u_A) + (B_1^{-1}u_B, u_B) - ((A_1+B_1)^{-1}u_A+u_B, u_A+u_B)\end{aligned}$$

and we may apply (2) to complete the proof of (4).

To prove Theorem 4 let $x_1, \cdots, x_m$ be an orthonormal set of eigenvectors of H corresponding respectively to $h_1, \cdots, h_m$. Let $y_1, \cdots, y_m$ be an orthonormal set of vectors in the orthogonal complement of the space spanned by $x_1, \cdots, x_m$ (possible since $m \leqq n/2$). Then using a result due to Fan [1, p. 114], we have from (2)

$$\begin{aligned}\sum_{j=1}^{m}\left(\frac{1}{h_j}+\frac{1}{k_{n-j+1}}\right) &\geqq \sum_{j=1}^{m}(H^{-1}x_j, x_j) + \sum_{j=1}^{m}(K^{-1}y_j, y_j)\\ &\geqq \sum_{j=1}^{m}((H+K)^{-1}x_j+y_j, x_j+y_j) \qquad (12)\\ &= 2\sum_{j=1}^{m}((H+K)^{-1}(x_j+y_j)/2^{1/2}, (x_j+y_j)/2^{1/2}).\end{aligned}$$

Now clearly $((x_j+y_j)/2^{1/2}, (x_k+y_k)/2^{1/2})=\delta_{jk}$ so that applying Fan's result again to the right side of (12) yields (5).

3. **An example.** Since (1) holds for $p\geqq 1$ it is plausible to conjecture that under the same hypotheses as Theorem 2, one has

$$(H^{-p}x, x) + (K^{-p}y, y) \geqq ((H+K)^{-p}x+y, x+y), \qquad (13)$$

for $p\geqq 1$. However, (13) is false in general for $p\geqq 1$. In particular let $p=2$, $y=0$, $u=(H+K)^{-1}x$ and the statement (13) becomes

$$\|u\|^2 \leqq \|(I+H^{-1}K)u\|^2. \qquad (14)$$

Now (14) is a possibility for all u if and only if the minimum singular value [1, p. 69] of $I+H^{-1}K$ is at least 1. At this point we use the following elementary result:

An n-square matrix A is the product of two positive-definite hermitian *matrices if and only if it has positive eigenvalues and linear elementary divisors.* For if $A=PQ$ where P and Q are positive-definite then $P^{-1/2}AP^{1/2}=P^{1/2}QP^{1/2}$ which is conjunctive to Q, and hence has positive eigenvalues and linear elementary divisors. But A is similar to $P^{1/2}QP^{1/2}$. Conversely, if A has linear elementary divisors and positive eigenvalues, then $A=S^{-1}DS$ in which D is a diagonal matrix with positive main diagonal entries. Let $S=UH$ be the polar decomposition of S so that

$$A = H^{-1}U^*DUH = H^{-2}H(U^*DU)H.$$

Then both H^{-2} and $H(U^*DU)H$ are positive-definite.

Thus we know for example that the matrix

$$A = \begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix}$$

is of the form $H^{-1}K$ for appropriate positive-definite H and K. It is elementary to compute that in this case the minimum singular value of $I+A$ is less than 1 and hence (14) is not true for all u.

We mention that in case H and K commute then (13) does hold for $p\geqq 1$. This is an easy consequence of the fact that H and K possess a common orthonormal basis of eigenvectors.

Reference

1. Marvin Marcus and Henryk Minc, *A survey of matrix theory and matrix inequalities*, Ginn, Boston, 1964.

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