ON PRODUCTS OF FINITE DIMENSIONAL STOCHASTIC MATRICES[1]

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1. In what follows, P is the monoid of all $p \times p$ stochastic matrices where p is a fixed natural number. For $a, a' \in P$, we let βa denote the "type" of a [1], i.e. the set of all pairs of indices (j, j') such that the element $a_{j,j'}$ of a is positive and we set $\|a - a'\| = \mathrm{Max}_j \sum_{j'} |a_{j,j'} - a'_{j,j'}|$. Letting ϵ and ω be two fixed positive quantities and $P(\omega)$ be the subset of all $a \in P$ having no positive element less than ω, we intend to verify the following partial generalization of a theorem of Wolfowitz [1].

PROPERTY. *There exists a natural number ν_* such that any product of more than ν_* matrices of $P(\omega)$ admits at least one nontrivial subproduct a which satisfies*

$$\beta a = \beta a^2 \quad \textit{and} \quad \sup_{n,n' \in N} \|a^{1+n} - a^{1+n'}\| \leqq \epsilon.$$

Our number ν_* is quite extravagant and examples such as $\lim_{n \to \infty} \prod_{0 \leq i < n} (x^{m_i} y)$ where the integers m_i grow fast enough,

$$x = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

indicate that little information on an infinite product of stochastic matrices is gained when knowing that for each positive ϵ it admits an infinity of subproducts a $(= x^{m_i})$ satisfying the relations stated in the Property (cf. [2]).

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2. **Verification of the property.** We say that two products $x_1 x_2 \cdots x_n$ and $x'_1 x'_2 \cdots x'_{n'}$ of matrices x_i, x'_i are *β-equivalent* iff $n = n'$ and $\beta x_i = \beta x'_i$ for $i = 1, 2, \cdots, n$. They are *nontrivial* iff $n > 0$.

Let $q = (2^p - 1)^p$ $(= \mathrm{Card}\ \beta P)$ and define inductively a map

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$\nu: \boldsymbol{N} \to \boldsymbol{N}$ and a sequence $(n_0, n_1, \cdots, n_q)$ of natural numbers by the following conditions:

$\nu(0)=1$; for each $n \in \boldsymbol{N}$, $\nu(n+1)=(1+q^{\nu(n)})\cdot\nu(n)$.

$n_0=1$; for each $i \in N$, $n_{i+1}=1+n_i+$the least positive number m such that $1-\omega^{\nu(ni)}$ to the power 2^m-1 is $\leqq\epsilon/20$.

REMARK 1. *Any product of $\nu_*=\nu(n_q)$ matrices of $P(\omega)$ admits a subproduct $a=h_1h_1'h_2h_2'h_3 \cdots h_{2s+1}h_{2s+1}'$ where*

(i) *all the $2s+1$ subproducts h_i are β-equivalent products of s' matrices of $P(\omega)$ and $(1-\omega^{s'})^s \leqq \epsilon/20$;*

(ii) *$\beta h_1=\beta a\cdot\beta h_1$ and $\beta a=\beta a^2$.*

PROOF. Call 0-*sesquipower* any nontrivial product and, inductively, say that a product is a $(n+1)$-*sesquipower* iff it has the form $hh'h''$ where h' is arbitrary and where h and h'' are β-equivalent n-sesquipowers.

We verify first that any product of $\nu(n)$ matrices admits at least one n-sesquipower as a subproduct.

Indeed, it is trivial for $n=0$ since $\nu(0)=1$. If it is true for n and if f is a product of $\nu(n+1)$ matrices, the definition of ν implies that $f=f_1f_2 \cdots f_{\bar q}$ $(\bar q=1+q^{\nu(n)})$ where each f_i is a product of $\nu(n)$ matrices. Since $q^{\nu(n)}$ is precisely the number of classes of β-equivalent product of $\nu(n)$ matrices, at least two of these subproducts (say f_j and $f_{j'}$) are β-equivalent. By the induction hypothesis we have $f_j=g_jhg_j'$; $f_{j'}=g_{j'}h''g_{j'}'$ where h and h'' are β-equivalent n-sesquipower and the statement is verified since f admits the subproduct $hh'h''$ where $h'=g_j'f_{j+1}f_{j+2} \cdots f_{j'-1}g_{j'}$.

In particular, if f is a product of ν_* matrices it admits a n_q-sesquipower k_q as a subproduct and for $j=q-1, q-2, \cdots, 0$; k_q admits a n_j-sesquipower k_j as a right subproduct. Again because of $q=\text{Card}\,\beta P$, at least two of these products (say $k_{j'}$ and k_j) have the same type. We can write $k_{j'}=h_1h_1'h_2 \cdots h_{2s+1}h_{2s+1}'k_j$ where $2s+2=2^{n_{j'}-n_j}$ and where all the subproducts h_i are β-equivalent to k_j. The conditions (i) of the Remark are automatically satisfied because of our choice of the subsequence (n_j) when we take $a=h_1h_1'h_2 \cdots h_{2s+1}h_{2s+1}'$ and we have $\beta h_1=\beta a\cdot\beta h_1$ since h_1, k_j and $k_{j'}=ak_j$ have the same image by β. The equation $\beta a=\beta a^2$ follows instantly from $\beta k_{j'}=\beta k_j=\beta h_1$ when multiplying on the right $k_{j'}=ak_j$ by $h_1'h_2 \cdots h_{2s+1}h_{2s+1}'$. The remark is verified.

REMARK 2. *For each given natural number n there exist a nonnegative quantity $\epsilon' \leqq \epsilon/10$ and two matrices $d, d' \in P$ that satisfy $a^{1+n}=(1-\epsilon')\cdot d+\epsilon'\cdot d'$ and $d=d\bar a d$ where $\bar a=\lim_{m\to\infty} a^m$.*

PROOF. We identify the indices $1, 2, \cdots, p$ with the states of the

Markov chain defined by the matrix a and we suppose that it has r ergodic classes $E_1, E_2, \cdots, E_r$.

For each $x \in P$, let

$$\pi x = \operatorname{Inf}\{\eta \in [0, 1] : x = (1 - \eta) \cdot x_r + \eta \cdot x_p; x_r \in \overline{P}_r; x_p \in P\}$$

where $\overline{P}_r$ (resp. $\overline{P}_p$) denotes the convex closure of the set P_r (resp. P_p) of all matrices $y \in P$ having entries 0 or 1 only and at most r (resp. p) nonzero columns. Thus, unless $\pi x = 1$, we have $1 - \pi x \geqq$ the least positive entry of x. Further, $\pi(xx') \leqq \pi x \cdot \pi x'$ for any $x, x' \in P$ since $P = \overline{P}_p$ and $P_r \subseteq P_p P_r P_p$ (cf. [3]).

In particular, $\pi a < 1$ because the relation $\beta a = \beta a^2$ implies that the type of any row of a contains at least one of the r ergodic classes. Taking $\beta k_{j'} = \beta k_j = \beta a \cdot \beta k_j$ into account, we deduce $\pi k_j < 1$, hence $\pi h_i < 1 - \omega^{s'}$ $(i = 1, 2, \cdots, 2s+1)$ since each h_i is a product of s' matrices of $P(\omega)$ that is β-equivalent to k_j.

Let us now define $b = a^n h_1 h_1' \cdots h_s h_s'$; $c = h_{s+1} h_{s+1}' \cdots h_{2s+1} h_{2s+1}'$; $d = b_r c_r$ $(b_r, c_r \in \overline{P}_r)$; $\epsilon' = \pi b + \pi c - \pi b \cdot \pi c$. Because of the submultiplicative character of the map π and $(1 - \omega^{s'})^s \leqq \epsilon/20$, we have $\pi b, \pi c \leqq \epsilon/20$, hence $\epsilon' \leqq \epsilon/10$ and $a^{1+n} = (1 - \epsilon') \cdot d + \epsilon' \cdot d'$ for a suitable $d' \in P$. Further, $\beta d \subseteq \beta a^{1+n} = \beta a$ and $\beta(d\bar{a}d) \subseteq \beta a$ because of $\beta \bar{a} \subseteq \beta a$. Since d and $d\bar{a}d$ are stochastic matrices, this shows that they have at least r characteristic roots equal to 1. To verify $d = d\bar{a}d$ we have only to check that the dimension of the null space of d has its maximal value $p - r$, since then it will follow that d and $d\bar{a}d$ are two commuting idempotent matrices having the same rank.

Consider any index i belonging to some ergodic class $E_{r'}$ $(1 \leqq r' \leqq r)$. Since $\beta(b_r c_r) \subseteq \beta a$ and since $E_{r'}$ is precisely the type of the ith row of a, we see that the type of the ith row of b_r must be contained in the set $E'_{r'}$ of the indices i' such that the type of the i'th row of c_r is contained in $E_{r'}$. The r sets $E'_{r'}$ are pairwise disjoint. Thus, since $b_r, c_r \in \overline{P}_r$, on the one hand the index of any nonzero column of b_r belongs to $\bigcup\{E'_{r'} : 1 \leqq r' \leqq r\}$ and, on the other hand, the type of any nonzero column of each $y \in P_r$ satisfying $\beta y \subseteq \beta c_r$ contains one (and only one) of the sets $E'_{r'}$.

Let $e'_{r'}$ be the diagonal matrix such that for any $j, j' = 1, 2, \cdots, p$, its (j, j') entry is equal to 1 or to 0 depending upon $j = j' \in E'_{r'}$ or not and $e' = e_1' + e_2' + \cdots + e_r'$. The first statement above implies that $d = b_r c_r = b_r e' c_r$ while the second one shows that all the nonzero rows of each matrix $e'_{r'} c_r$ are equal, hence that the null space of $e' c_r$ has dimension at least $p - r$. Remark 2 is verified.

Substituting $(1 - \epsilon') \cdot d + \epsilon' \cdot d'$ for a^{1+n} in the right member of

$a^{1+n}-\bar{a}=a^{1+n}-a^{1+n}\bar{a}a^{1+n}$ and recalling that $\|x\|=1$ for any $x\in P$, we obtain

$$\begin{aligned}\|a^{1+n}-\bar{a}\| &= \epsilon'\cdot\|(1-\epsilon')(d-d\bar{a}d'-d'\bar{a}d)+d'-\epsilon'\cdot d'\bar{a}d'\| \\ &\leqq 5\epsilon' \leqq \epsilon/2.\end{aligned}$$

In view of the triangular inequality, this concludes the verification of the Property.

References

1. J. Wolfowitz, *Products of indecomposable aperiodic stochastic matrices*, Proc. Amer. Math. Soc. **14** (1963), 733–737.

2. N. J. Pullman, *Infinite products of substochastic matrices*, Pacific J. Math. **16** (1966), 536–544.

3. J. Larisse et M. P. Schützenberger, *Sur certaines chaines de Markov non homogènes*, Publ. Inst. Statist. Univ. Paris **13** (1964), 57–66.

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