COMPLETENESS IN FINITE ALGEBRAS WITH A SINGLE OPERATION

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An algebra $\langle A, (\omega_i)_{i \in I} \rangle$ is said to be complete if for each n all functions $f: A^n \rightarrow A$ are definable in terms of the operations ω_i ($i \in I$). The purpose of this note is to show that a finite algebra $\langle A, \omega \rangle$ with a single operation is complete if and only if it has no subalgebras, automorphisms or homomorphic images apart from the obvious trivial ones. This generalizes results of Post [1] ($|A| = 2$) and Wheeler [2] ($|A| = 3$). The proof is based on a theorem of Rosenberg [3] which characterizes complete finite algebras with any number of operations; the latter theorem in turn generalizes results of Post [1] ($|A| = 2$) and Jablonskiĭ [4] ($|A| = 3$).¹

As a corollary of the main result we show that a finite algebra $\langle A, \omega \rangle$ with more than two elements is complete iff ω generates a doubly transitive subgroup of the symmetric group on A . This improves results of Salomaa [5] and Schofield [7].

In order to state Rosenberg's theorem we need to introduce certain definitions. For $k > 0$ we denote by E_k the set $\{0, 1, \dots, k-1\}$. An h -ary relation is a subset R of E_k^h , and is called universal if $R = E_k^h$; we write $R(x_1, \dots, x_h)$ when $(x_1, \dots, x_h) \in R$. If f is a function of n arguments and R is an h -ary relation we say that f preserves R if $R(f(x_1^1, \dots, x_1^n), \dots, f(x_h^1, \dots, x_h^n))$ whenever $R(x_1^1, \dots, x_h^1), \dots, R(x_1^n, \dots, x_h^n)$.

An h -ary relation is totally reflexive if $R(x_1, \dots, x_h)$ whenever $x_1, \dots, x_h$ are not all distinct, and totally symmetric if for each permutation σ of the integers $1, \dots, h$, $R(x_1, \dots, x_h) \Leftrightarrow R(x_{\sigma(1)}, \dots, x_{\sigma(h)})$. The centre of a totally symmetric h -ary relation is the set of elements c such that $R(x_1, \dots, x_{h-1}, c)$ holds for all $x_1, \dots, x_{h-1} \in E_k$. A relation is said to be central if it is totally reflexive and totally symmetric and has centre C , $\emptyset \subset C \subset E_k$. Thus the singularly central relations are just the proper nonnull subsets of E_k . For $a \in E_k^m$ we denote by $[a]_l$ the l th digit ($l = 0, \dots, m-1$) in the expansion $a = \sum_{l=0}^{m-1} [a]_l h^l$ of a in the scale of h .

We may now state the theorem of Rosenberg as follows:

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¹ Dr. Roy O. Davies kindly communicated to me a derivation of Wheeler's result from the theorem of Jablonskiĭ and remarked that he had established Theorem 1 for $|A| \leq 8$ using Rosenberg's theorem (the proof of which has not yet been published).

For a finite algebra $\langle E_k, (\omega_i)_{i \in I} \rangle$ ($k > 2$) to be complete it is necessary and sufficient that in each of the following cases there exists at least one ω_i ($i \in I$) that does not preserve the relation R :

- 1°. R is a partial order on E_k with greatest and least element.
- 2°. R is (the graph of) a nonidentical permutation of E_k .
- 3°. R is the relation $\langle\langle x_1 + x_2 = x_3 + x_4 \rangle\rangle$ where $k = p^m$ and $G = \langle E_k, + \rangle$ is an abelian group in which each nonzero element has order p .
- 4°. R is a nonidentical, nonuniversal equivalence relation on E_k .
- 5°. R is an h -ary central relation on E_k ($1 \leq h \leq k$).
- 6°. R is for some h ($3 \leq h \leq k$) and some surjection $\phi: E_k \rightarrow E_h^m$ ($m > 0$) the h -ary relation

$$| \{ [\phi(x_1)]_l, \dots, [\phi(x_h)]_l \} | < h \quad \text{for } l = 0, \dots, m-1$$

(i.e., for each l ($l = 0, \dots, m-1$) at least two of the numbers $\phi(x_1), \dots, \phi(x_h)$ have the same l th digit when expressed in the scale of h).

If $\langle A, (\omega_i)_{i \in I} \rangle$ is an arbitrary algebra, then a nonempty subset of A , a permutation of A , or an equivalence on A is called a subalgebra, automorphism or congruence respectively iff it is preserved by each ω_i ($i \in I$). A subalgebra distinct from A is said to be proper, as is an automorphism other than the identity, or a nonuniversal, nonidentical congruence. The existence of proper congruences on A is equivalent to the existence of homomorphic images of A not isomorphic to A or to the one-element algebra.

THEOREM 1. *A finite algebra $\langle A, \omega \rangle$ with a single operation is complete if and only if it has no proper subalgebras, automorphisms or congruences.*

PROOF. In view of the above-mentioned result of Post we may restrict attention to the case $A = E_k$, $k > 2$. The necessity of the condition is clear from 2°, 4°, 5° of Rosenberg's theorem. To prove sufficiency we have to show in cases 1°–6° that if ω preserves R then A has a proper subalgebra, automorphism or congruence.

1°. If ω preserves a partial order $\leq$ with greatest element u , then either $\{u\}$ or $A \setminus \{u\}$ is a proper subalgebra. For, if $\omega(x_1, \dots, x_n) = u$ for some $x_1, \dots, x_n \in A \setminus \{u\}$ then, because $x_1 \leq u, \dots, x_n \leq u$, we have $u = \omega(x_1, \dots, x_n) \leq \omega(u, \dots, u)$ so that $\omega(u, \dots, u) = u$.

3°. Suppose $k = p^m$ and $G = \langle E_k, + \rangle$ is an abelian group in which each nonzero element has order p . Then G , being the direct sum of m cyclic groups of order p , may be identified with the space $(Z_p)^m$ where Z_p is the residue class field of the integers modulo p . If ω preserves the relation $\langle\langle x_1 + x_2 = x_3 + x_4 \rangle\rangle$ then it satisfies the functional equation

$$\omega(x_1, \dots, x_n) + \omega(y_1, \dots, y_n) = \omega(x_1 + y_1, \dots, x_n + y_n) + \omega(0, \dots, 0)$$

and it is not difficult to show that the solutions are of the form

$$\omega(x_1, \dots, x_n) = a + x_1 A_1 + \dots + x_n A_n$$

where $a \in (Z_p)^m$ and $A_1, \dots, A_n$ are $m \times m$ matrices over Z_p . If $\det(I - \sum A_i) = 0$ then the equation $0 = x(I - \sum A_i)$ has a nonzero solution $x = x_0$, so that

$$\omega(x_0 + x_1, \dots, x_0 + x_n) = x_0 + \omega(x_1, \dots, x_n);$$

i.e. $x \rightarrow x_0 + x$ is a proper automorphism. If $\det(I - \sum A_i) \neq 0$ then the equation $a = x(I - \sum A_i)$ has a solution $x = x_0$, so that $\omega(x_0, \dots, x_0) = x_0$; i.e. $\{x_0\}$ is a proper subalgebra.

5°. Suppose ω preserves the h -ary central relation R . Let C_0 be the centre of R and for each ν let $C_{\nu+1} = C_\nu \cup \omega(C_\nu^n)$; we show that $C_\nu^n \subseteq R$ for each ν . This is clear for $\nu = 0$; suppose $C_\nu^n \subseteq R$, $(x_1, \dots, x_h) \in C_{\nu+1}^h$. Then $(x_1, \dots, x_h) \in R$ if some $x_i \in C_0$. If no $x_i \in C_0$ we may write $x_i = \omega(x_i^1, \dots, x_i^n)$ with $(x_i^1, \dots, x_i^n) \in C_\nu^n$ ($i = 1, \dots, h$), so again $(x_1, \dots, x_h) \in R$ since by induction hypothesis $R(x_1^1, \dots, x_h^1), \dots, R(x_1^n, \dots, x_h^n)$. Since R is nonuniversal it follows from what has been proved that $\bigcup_{\nu \geq 0} C_\nu \neq E_k$; thus $\bigcup_{\nu \geq 0} C_\nu$ is a proper subalgebra.

6°. Suppose ω preserves the h -ary relation R , where $3 \leq h \leq k$ and for some surjection $\phi: E_k \rightarrow E_{h^m}$ ($m > 0$), $R(x_1, \dots, x_h)$ is equivalent to

$$|\{[\phi(x_1)]_l, \dots, [\phi(x_h)]_l\}| < h \quad \text{for } l = 0, \dots, m-1.$$

(a) If $h^m < k$ then $\langle\langle \phi(x_1) = \phi(x_2) \rangle\rangle$ is a proper congruence unless E_k has a proper subalgebra. For, $\langle\langle \phi(x_1) = \phi(x_2) \rangle\rangle$ is a nonidentical, nonuniversal equivalence relation; we show that $\phi(x_1^1) = \phi(x_2^1), \dots, \phi(x_1^n) = \phi(x_2^n)$ imply $\phi(\omega(x_1^1, \dots, x_1^n)) = \phi(\omega(x_2^1, \dots, x_2^n))$. Suppose the contrary. For some l ($l = 0, \dots, m-1$), $[\phi(\omega(x_1^1, \dots, x_1^n))]_l \neq [\phi(\omega(x_2^1, \dots, x_2^n))]_l$. If E_k has no proper subalgebra the mapping $\omega: E_k^n \rightarrow E_k$ is surjective, as are $\phi: E_k \rightarrow E_{h^m}$ and $[\dots]_l: E_{h^m} \rightarrow E_h$. Thus elements $x_i^1, \dots, x_i^n$ ($i = 3, \dots, h$) may be chosen so that $[\phi(\omega(x_i^1, \dots, x_i^n))]_l$ ($i = 1, \dots, h$) are all distinct, i.e. $R(\omega(x_1^1, \dots, x_1^n), \dots, \omega(x_h^1, \dots, x_h^n))$ does not hold. Since $R(x_1^1, \dots, x_h^1), \dots, R(x_1^n, \dots, x_h^n)$ hold, this contradicts the supposition that ω preserves R .

(b) Suppose $h^m = k$ so that ϕ is bijective. Since a function $f(x^1, \dots, x^n)$ preserves a relation $S(y_1, \dots, y_h)$ iff $\phi(f(\phi^{-1}(x^1), \dots, \phi^{-1}(x^n)))$ preserves $S(\phi^{-1}(y_1), \dots, \phi^{-1}(y_h))$ it is easy to see that it suffices to consider the case where ϕ is the identity: i.e. R is the relation

$$|\{[x_1]_l, \dots, [x_h]_l\}| < h \quad \text{for } l = 0, \dots, m-1.$$

Consider the mappings $g_l: E_h^{mn} \rightarrow E_h$ ($l=0, \dots, m-1$) defined by

$$\begin{aligned} g_l(\xi_0^1, \dots, \xi_{m-1}^1; \dots; \xi_0^n, \dots, \xi_{m-1}^n) \\ = \left[\omega \left(\sum_{j=0}^{m-1} \xi_j^1 h^j, \dots, \sum_{j=0}^{m-1} \xi_j^n h^j \right) \right]_l. \end{aligned}$$

We may suppose each g_l surjective for otherwise E_k has a proper subalgebra. If some g_l depends essentially on more than one of its arguments then (cf. [4], [6], [8]) we can find $(h-1)$ -element subsets $H_1, \dots, H_{mn}$ of E_h such that g_l assumes h values on $H_1 \times \dots \times H_{mn}$; but this would imply that ω does not preserve R . Thus each g_l ($l=0, \dots, m-1$) depends essentially on only one argument, so we may write

$$g_l(\xi_0^1, \dots, \xi_{m-1}^1; \dots; \xi_0^n, \dots, \xi_{m-1}^n) = \gamma_l(\xi_{\tau(l)}^{\sigma(l)}) \quad (l = 0, \dots, m-1)$$

and

$$\omega(x^1, \dots, x^n) = \sum_{l=0}^{m-1} \gamma_l([x^{\sigma(l)}]_{\tau(l)}) h^l.$$

If some proper nonnull subset I of the indices $0, \dots, m-1$ is invariant under the mapping τ ($\tau(I) \subseteq I$), then the relation $\langle\langle [x]_i = [y]_i \text{ for each } i \in I \rangle\rangle$ is a proper congruence. In the contrary case τ is a permutation of $0, \dots, m-1$ consisting of a single cycle of length m . Set

$$\alpha(x) = \omega(x, \dots, x) = \sum_{l=0}^{m-1} \gamma_l([x]_{\tau(l)}) h^l.$$

By induction we have

$$\alpha^r(x) = \sum_{l=0}^{m-1} \gamma_l \gamma_{\tau(l)} \dots \gamma_{\tau^{r-1}(l)}([x]_{\tau^r(l)}) h^l,$$

so, in particular,

$$\alpha^m(x) = \sum_{l=0}^{m-1} \gamma_l \gamma_{\tau(l)} \dots \gamma_{\tau^{m-1}(l)}([x]_l) h^l.$$

From this we verify that

$$\alpha^m(\omega(x^1, \dots, x^n)) = \omega(\alpha^m(x^1), \dots, \alpha^m(x^n)).$$

If α^m is not the identity we consider three cases: (i) if α^m is constant $= x_0$, then

$$\omega(x_0, \dots, x_0) = \omega(\alpha^m(x_0), \dots, \alpha^m(x_0)) = \alpha^m(\omega(x_0, \dots, x_0)) = x_0$$

so that $\{x_0\}$ is a proper subalgebra; (ii) if α^m is one-one then α^m is a proper automorphism; (iii) if α^m is not constant and not one-one, then $\langle\langle\alpha^m(x_1) = \alpha^m(x_2)\rangle\rangle$ is a proper congruence. If α^m is the identity then $\gamma_l \gamma_{r(l)} \dots \gamma_{r^{m-1}(l)}$ is the identity on E_h ($l=0, \dots, m-1$). Choosing $\xi_0 \in E_h$ arbitrarily we may set

$$\xi_{r(0)} = \gamma_{r(0)} \dots \gamma_{r^{m-1}(0)}(\xi_0) \quad (r = 0, \dots, m-1),$$

so that $\gamma_l(\xi_{r(l)}) = \xi_l$ ($l=0, \dots, m-1$); then if $x = \sum_{i=0}^{m-1} \xi_i h^i$ we have $\omega(x, \dots, x) = x$, i.e. $\{x\}$ is a proper subalgebra.

The proof of Theorem 1 is now complete.

THEOREM 2. *A finite algebra $\langle A, \omega \rangle$ with more than two elements is complete if and only if the set of functions definable from ω includes a doubly transitive subgroup of the symmetric group on A .*

PROOF. If A is not complete then by Theorem 1 it has a proper subalgebra, automorphism or congruence, which must be preserved by all functions definable in terms of ω . But in general if $R(a_1, \dots, a_h)$ holds while $R(a'_1, \dots, a'_h)$ does not, where $\{a_1, \dots, a_h\}$ and $\{a'_1, \dots, a'_h\}$ each consist of h elements, then R is not preserved by any permutation which carries a_1 to $a'_1, \dots, a_h$ to a'_h . Applying this in the case of a proper subalgebra, automorphism or congruence, where h does not exceed 2, it follows that ω cannot generate a doubly transitive group of permutations.

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